Chapter 2

Higher-order equations, first-order system

2.1 Second-order differential equation

The general form of a second-order differential equation is:

$$F(t, y, y', y'') = 0$$

or in the normal form: $Y'' = F(t, y, y')$.

The general solution "Y" usually depends on two parameters, λ and μ.

2.1.1 Second-order linear differential equations

Consider the second-order differential equation:

$$Y'' + a(t)Y' + b(t)Y = c(t) \tag{2.1}$$

(a) If $c(t) = 0$, equation (2.1) is a homogeneous differential equation.

(b) If

$$\begin{cases} a(t) = a \\ b(t) = b \end{cases}$$

we obtain $Y'' + aY' + bY = c(t)$, which is a linear equation with constant coefficients.

Theorem 2.1.1.

The general solution of a second-order differential equation is given by:

$$S_G = S_H + S_P$$

S_H: Solution of the homogeneous equation.

S_P: A particular solution.

Definition 2.1.

Two solutions y_1 and y_2 of equation (2.1) are independent on an interval I if there is no real number k such that:

$$\text{for all } t \in I: \; y_2(t) = k y_1(t).$$

Remark 2.1.1.

The two functions y_1 and y_2 are independent, i.e., they are linearly independent in the sense of vector spaces.

Definition 2.2.

Let two functions be differentiable on the interval I.

y_1, y_2 are linearly independent if and only if the determinant

$$\begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}$$

is not identically zero.

Example 2.1.1.

The functions $\sin t$ and $\cos t$ are independent:

$$\begin{vmatrix} \sin t & \cos t \\ \cos t & -\sin t \end{vmatrix} = -\sin^2 t - \cos^2 t = -(\sin^2 t + \cos^2 t) = -1 \neq 0 \quad \forall t.$$

2.1.2 Second-order homogeneous linear equations

Case Where Two Independent Particular Solutions Are Known

If y_1, y_2 are two independent solutions of the equation:

$$y'' + a(t)y' + b(t)y = 0$$

the general solution is given by:

$$y = \lambda y_1 + \mu y_2,$$

where λ, μ are constants.

Case Where One Particular Solution Is Known

If a particular solution y_1 is known, we set the change of variable $y(t) = Y_1(t)v(t)$. $y'(t) = Y_1'v + v'Y_1, \quad Y''(t) = Y_1''v + v'Y_1' + v''Y_1 + Y_1'v' = Y_1''v + 2Y_1'v' + v''Y_1$. Substituting Y, Y', Y'' into the homogeneous equation, we obtain a second-order differential equation for the unknown v:

$$y_1 v'' + (2y_1' + a(t)y_1)v' = 0$$

We make the second change of variable "$w = v'$", giving:

$$\begin{aligned}
\frac{w'}{w} = \frac{v''}{v'} &= -2 \cdot \frac{y_1'}{y_1} - a(t) \\
\int \frac{dw}{w} &= -2 \int \frac{Y_1'}{Y_1} dt - \int a(t)dt \\
\ln w &= -2 \ln Y_1 - \int a(t)dt + c \\
w &= \frac{1}{Y_1^{-2}} \cdot e^{-\int a(t)dt} \cdot \lambda, \qquad \lambda \in \mathbb{R} \\
w(t) &= \lambda \cdot \frac{1}{Y_1^{-2}} \cdot e^{-\int a(t)dt}.
\end{aligned}$$

Hence:

$$v(t) = \int w(t)dt + \mu, \qquad \mu \in \mathbb{R}.$$

The general solution of the equation:

$$y'' + a(t)y' + b(t)y = 0$$

is therefore:

$$\begin{aligned} y(t) &= y_1(t)v(t) \\ &= y_1(t)\Big(\int w(t)dt + \mu\Big) = y_1(t)\int w(t)dt + \mu y_1(t), \qquad \mu \in \mathbb{R}. \end{aligned}$$

Consider the equation:

$$(t+1)y'' - (2t-1)y' + (t-2)y = 0$$

We can verify that $y_1(t) = e^t$ is a particular solution.

Let us seek the general solution in the form $y(t) = e^t v(t)$:

$$\begin{aligned} y'(t) &= e^t v + v'e^t \\ y''(t) &= e^t v + v'e^t + v''e^t + e^t v' = e^t v + 2v'e^t + e^t v''. \end{aligned}$$

Substituting y, y', y'' into the homogeneous equation, we get:

$$\begin{aligned} (1+t)[e^t(v + 2v' + v'')] + (2t-1)[e^t(v+v')] + (t-2)e^t v &= 0 \\ e^t[(t+1)v + 2(1+t)v' + (1+t)v'' - (2t-1)v - (2t-1)v' + (t-2)v] &= 0 \\ e^t[(t+1)v'' + v'[2(t+1) - (2t-1)] + v[(t-1) - (2t-1) + (t-2)]] &= 0 \\ e^t[(t+1)v'' + 3v' + 0\cdot v] &= 0, \end{aligned}$$

Since $e^t \neq 0$ for all $t \in \mathbb{R}$:

$$(t+1)v'' + 3v' = 0 \Longrightarrow \frac{v''}{v'} = \frac{-3}{1+t}, \qquad t \neq -1.$$

Let $\begin{cases} w = v' \\ w' = v'' \end{cases}$, we find:

$$\begin{aligned} \frac{w'}{w} = \frac{-3}{1+t} &\Longrightarrow \int \frac{dw}{w} = \int \frac{-3}{1+t}dt \\ \ln|w| &= -3\ln(1+t) + c \\ w = e^c \frac{1}{(1+t)^3} &\Longrightarrow w = \frac{\lambda}{(1+t)^3}, \quad \lambda \in \mathbb{R}. \end{aligned}$$

Hence:

$$\begin{aligned} v' &= \int \frac{\lambda}{(1+t)^3}dt = \lambda\Big(\frac{-1}{2}\cdot\frac{1}{(1+t)^2}\Big) + \mu \\ v(t) &= \frac{\lambda}{(1+t)^2} + \mu, \qquad \lambda, \mu \in \mathbb{R}. \end{aligned}$$

The general solution is given by:

$$y(t) = e^t v(t) = \lambda \frac{e^t}{(1+t)^2} + \mu e^t, \qquad \lambda, \mu \in \mathbb{R}.$$

2.2 Second-order non-homogeneous differential equation with non-constant coefficients

Consider the equation:

$$y'' + a(t)y' + b(t)y = c(t), \qquad c(t) \neq 0.$$

As for first-order linear equations.

Theorem 2.2.1.

The general solution of the non-homogeneous equation:

$$y'' + a(t)y' + b(t)y = c(t)$$

is equal to the sum of the general solution of the homogeneous equation and a particular solution of the non-homogeneous equation:

$$S_G = S_H + S_P$$

S_G: The general solution.

S_H: Solution of the homogeneous equation.

S_P: A particular solution.

2.2.1 Method of Variation of Parameters

The principle of this method is to consider λ and μ as functions of the variable t: Suppose we seek the solution in the form:

$$y_p(t) = \lambda(t)y_1(t) + \mu(t)y_2(t).$$

Explanation:

Substituting this function into the non-homogeneous equation, after simplification, we get:

$$\begin{aligned} &b(y = \lambda y_1 + \mu y_2) \\ &ay' = (\lambda' y_1 + y_1'\lambda + \mu' y_2 + y_2'\mu)a \\ &\ y'' = \lambda'' y_1 + y_1'\lambda' + y_1''\lambda + \lambda' y_1' + \mu'' y_2 + y_2'\mu' + y_2''\mu + \mu' y_2' \\ &\quad\ = \lambda'' y_1 + \lambda'(2y_1') + \lambda y_1'' + \mu'' y_2 + \mu'(2y_2') + \mu y_2''. \end{aligned}$$

Thus:

$$\begin{aligned} &\lambda'' y_1 + \lambda'(2y_1') + \lambda y_1'' + \mu'' y_2 + \mu'(2y_2') + \mu y_2'' + a\lambda' y_1 + a y_1'\lambda + a\mu' y_2 + a y_2'\mu + b y_1\lambda + b y_2\mu \\ &\ = \lambda'' y_1 + \mu'' y_2 + \lambda'(2y_1' + a y_1) + \mu'(2y_2' + a y_2) + \lambda(y_1'' + a y_1' + b y_1) + \mu(y_2'' + a y_2' + b y_2) = c(t) \\ &\lambda'' y_1 + \mu'' y_2 + \lambda'(2y_1' + a y_1) + \mu'(2y_2' + a y_2) = c(t). \end{aligned}$$

Using the method of variation of constants such that:

$$\lambda' y_1 + \mu' y_2 = 0, \quad \lambda' y_1' + \mu' y_2' = c(t).$$

Solving this system, we obtain:

$$\lambda' = \frac{-c(t)y_2(t)}{y_1(t)y_2'(t) - y_1'(t)y_2(t)}, \quad \mu' = \frac{+c(t)y_1(t)}{y_1(t)y_2'(t) - y_1'(t)y_2(t)}.$$

Hence, the particular solution is given by:

$$y_p(t) = y_1(t)\int \frac{-c(t)Y_2}{Y_1Y_2' - Y_1'Y_2}dt + y_2(t)\int \frac{c(t)Y_1}{Y_1Y_2' - Y_1'Y_2}dt.$$

Consider the equation:

$$y'' - \frac{2}{t^2}y = te^t, \quad y = t^\alpha, \quad \alpha \in \mathbb{R}, \quad t \in]0, +\infty[$$

We first seek solutions for the homogeneous equation:

$$y'' - \frac{2}{t^2}y = 0 \text{ in the form } y(t) = t^\alpha, \quad \alpha \in \mathbb{R}.$$

We have:

$$\begin{aligned} y = t^\alpha &\Longrightarrow \frac{2}{t^2}y = 2t^{\alpha-2}, \\ y' &= \alpha t^{\alpha-1}, \\ y'' &= \alpha(\alpha-1)t^{\alpha-2}, \end{aligned}$$

Summing these terms, we find:

$$\begin{aligned} y'' - \frac{2}{t^2}y = t^{\alpha-2}[\alpha(\alpha-1) - 2] = 0 &\Longrightarrow \alpha(\alpha-1) - 2 = 0 \\ &\Longrightarrow \alpha^2 - \alpha - 2 = 0 \end{aligned}$$

$$\text{Discriminant: } \Delta = 1 - 4(-2) = 9, \quad \text{thus: } \alpha_1 = -1, \quad \alpha_2 = 2.$$

- For $\alpha_1 = -1 \Longrightarrow y_1(t) = \frac{1}{t}$.
- For $\alpha_2 = 2 \Longrightarrow y_2(t) = t^2$.

Hence, the general solution of the homogeneous equation is:

$$y_H(t) = \lambda y_1 + \mu y_2 = \lambda \cdot t^2 + \mu \cdot \frac{1}{t}, \qquad \lambda, \mu \in \mathbb{R}.$$

Assume the particular solution is given by:

$$y_p(t) = \frac{\lambda(t)}{t} + \mu(t)t^2, \quad \text{with derivatives } \lambda', \mu' \text{ satisfying: } \begin{cases} \lambda' y_1 + \mu' y_2 = 0 \\ \lambda' y_1' + \mu' y_2' = te^t \end{cases}$$

Substituting $y_1 = 1/t$, $y_2 = t^2$:

$$\begin{cases} \lambda' \frac{1}{t} + \mu' t^2 = 0 \\ \lambda'(-\frac{1}{t^2}) + \mu'(2t) = te^t \end{cases}$$

Using Cramer's rule (since $\det \neq 0$):

$$\begin{bmatrix} 1/t & t^2 \\ -1/t^2 & 2t \end{bmatrix} = 3 \neq 0$$

$$\lambda' = \frac{\begin{bmatrix} 0 & t^2 \\ te^t & 2t \end{bmatrix}}{3} = \frac{t^3e^t}{3} \Longrightarrow \lambda(t) = \frac{1}{3}e^t(-t^3+3t^2-6t+6),$$
$$\mu' = \frac{\begin{bmatrix} 1/t & 0 \\ -1/t^2 & te^t \end{bmatrix}}{3} = \frac{e^t}{3} \Longrightarrow \mu(t) = \frac{1}{3}e^t.$$

Hence:

$$\begin{aligned} S_p(t) &= \frac{1}{t}\left(\frac{1}{3}e^t(-t^3+3t^2-6t+6)\right) + t^2\left(\frac{1}{3}e^t\right) \\ &= \frac{1}{3}e^t(-t^2+3t-6+6/t+t^2) \\ &= \frac{1}{3}e^t(3t-6+6/t) \\ &= e^t(t-2+2/t) \end{aligned}$$

Thus, the general solution is:

$$S_G(t) = \lambda \cdot (1/t) + \mu \cdot t^2 + e^t(t-2+2/t).$$

2.3 Second-order linear equation with constant coefficients

A second-order linear differential equation with constant coefficients (2nd-order LDE with constant coefficients) is an equation of the form:

$$y'' + ay' + by = c(t)$$

where a, b are real constants, and $t \mapsto c(t)$ is a given continuous function on an interval $I \subset \mathbb{R}$.

We start by solving the associated homogeneous equation:

$$y'' + ay' + by = 0 \tag{2.2}$$

We look for solutions of the form:

$$y = e^{rt}, \qquad r \in \mathbb{R}.$$

Substituting into the homogeneous equation (2.2), we get:

$$(r^2 + ar + b)e^{rt} = 0, \quad \text{since } e^{rt} \neq 0, \forall t \in \mathbb{R}.$$

For a solution to exist, we must have:

$$r^2 + ar + b = 0.$$

This equation is called the **characteristic equation** associated with the homogeneous equation (2.2).

We have three cases:

(a) If $a^2 - 4b > 0$, we find two distinct real roots r_1 and r_2. The general solution of the homogeneous equation is then:

$$y_H = \lambda e^{r_1 t} + \mu e^{r_2 t}, \qquad \lambda, \mu \in \mathbb{R}.$$

(b) If $a^2 - 4b = 0$, we find a double real root r_0. The general solution of the homogeneous equation is:

$$y_H = (\lambda t + \mu)e^{r_0 t}, \qquad \lambda, \mu \in \mathbb{R}.$$

(c) If $a^2 - 4b < 0$, we find two distinct complex conjugate roots of the form:

$$\begin{cases} r_1 = \alpha - i\beta \\ r_2 = \alpha + i\beta \end{cases}, \qquad \alpha, \beta \in \mathbb{R}.$$

The general solution of the homogeneous equation is:

$$y_H = e^{\alpha t}(\lambda \cos(\beta t) + \mu \sin(\beta t)), \qquad \alpha, \beta, \lambda, \mu \in \mathbb{R}.$$

Example 2.3.1.

Consider the following equation:

$$y'' + 4y' + 3y = 0$$

The characteristic equation (C.E.) is: $r^2 + 4r + 3 = 0.$

We have: $r_1 = -3$ and $r_2 = -1$, so the general solution is:

$$y = \lambda e^{-3t} + \mu e^{-t}, \qquad \lambda, \mu \in \mathbb{R}.$$

2.3.1 Search for a Particular Solution for Specific Second-Order Differential Equations

If $c(t) = p(t)$ where $p(t)$ is **a polynomial** of degree "n".

- Look for a solution $q(t)$ that is a **polynomial** of degree:

(a) If $b \neq 0 \Longrightarrow q(t)$ of degree "n".

(b) If $b = 0, \quad a \neq 0 \Longrightarrow q(t)$ of degree "$n+1$".

(c) If $b = 0$ and $a = 0 \Longrightarrow q(t)$ of degree "$n+2$".

Example 2.3.2.

Consider the equation:

$$y'' - y' - 2y = 2t^2,$$

The characteristic equation (C.E.) is: $r^2 - r - 2 = 0$ which admits two real roots:

$$r_1 = -1 \text{ and } r_2 = 2$$

Hence, the general solution is:

$$S_G = \lambda e^{-t} + \mu e^{2t}, \qquad \lambda, \mu \in \mathbb{R}.$$

We look for a particular solution in the form of a 2nd-degree polynomial:

$$\begin{aligned} y_p(t) &= \alpha t^2 + \beta t + \gamma \\ y_p'(t) &= 2\alpha t + \beta \\ y_p''(t) &= 2\alpha \end{aligned}$$

The equation becomes:

$$\begin{aligned} y_p'' - y_p' - 2y_p = 2t^2 &\iff 2\alpha - (2\alpha t + \beta) - 2(\alpha t^2 + \beta t + \gamma) = 2t^2 \\ &\iff (2\alpha - \beta - 2\gamma) + t(-2\alpha - 2\beta) + t^2(-2\alpha) = 2t^2 \end{aligned}$$

By identification, we have:

$$\begin{cases} 2\alpha - \beta - 2\gamma = 0 \\ -2\alpha - 2\beta = 0 \\ -2\alpha = 2 \end{cases} \iff \begin{cases} \gamma = \dfrac{2\alpha - \beta}{2} = -\dfrac{3}{2} \\ \beta = 1 \\ \alpha = -1 \end{cases}$$

Hence, the general solution is:

$$y_G = \lambda e^{-t} + \mu e^{2t} + (-t^2 + t - \frac{3}{2}), \qquad \lambda, \mu \in \mathbb{R}.$$

2.4 Summary of the Possible Cases

Summarize the possible cases in the following table:

Second term $c(t)$	Particular solution $y_p(t)$
$c(t) = p_n(t)$ is a polynomial of degree "n"	$\begin{cases} \bullet y_p(t) = q_n(t), \quad \text{if} \quad b \neq 0. \\ \bullet y_p(t) = q_{n+1}(t), \quad \text{if} \quad b = 0 \quad \text{and} \quad a \neq 0. \\ \bullet y_p(t) = q_{n+2}(t), \quad \text{if} \quad b = 0 \quad \text{and} \quad a = 0. \end{cases}$
$c(t) = ke^{rt}$ with $r^2 + ar + b \neq 0$	$\bullet y_p(t) = \alpha e^{rt}; \quad r$ is not a root.
$c(t) = ke^{rt}$ with $r^2 + ar + b = 0$	$\begin{cases} \bullet y_p = \alpha t e^{rt}; \quad r \text{ simple root.} \\ \bullet y_p = \alpha t^2 e^{rt}; \quad r \text{ double root.} \end{cases}$
$c(t) = p_n(t)e^{rt}, \quad r^2 + ar + b \neq 0$	$\bullet y_p(t) = q_n(t)e^{rt}; \quad \deg(q) = n.$
$c(t) = p_n(t)e^{rt}, \quad r^2 + ar + b = 0$	$\bullet y_p(t) = q_{n+1}(t)e^{rt}; \quad \deg(q) = n + 1.$
$c(t) = p_n(t)e^{rt}; \quad r = \frac{-a}{2}$ and $\deg(p) = n$	$\bullet y_p(t) = q_{n+2}(t)e^{rt}; \quad \deg(q) = n + 2.$
$c(t) = d\cos(rt) + e\sin(rt)$	$\bullet y_p(t) = \alpha\cos(rt) + \beta\sin(rt).$

Example 2.4.1.

Consider the equation:

$$y'' - y' - 2y = \sin(2t).$$

The general solution is:

$$S_G = \lambda e^{-t} + \mu e^{2t}, \quad \lambda, \mu \in \mathbb{R}.$$

The particular solution is:

$$y_p = \alpha \cos(2t) + \beta \sin(2t),$$

with $\alpha = \frac{1}{20}$ and $\beta = -\frac{3}{20}$. Hence, the general solution is:

$$y_G = \frac{1}{20}\cos(2t) - \frac{3}{20}\sin(2t) + \lambda e^{-t} + \mu e^{2t}, \quad \lambda, \mu \in \mathbb{R}.$$

Example 2.4.2.

Consider the equation:

$$y'' - y' - 2y = te^t \Longrightarrow y_H = \lambda e^{-t} + \mu e^{2t}, \quad \lambda, \mu \in \mathbb{R}. \tag{2.3}$$

Since 1 is not a root of the characteristic equation, we look for a particular solution of the form:

$$y_p = (\alpha t + \beta)e^t.$$

We have:

$$y_p' = (\alpha t + \alpha + \beta)e^t,$$
$$y_p'' = (\alpha t + 2\alpha + \beta)e^t.$$

Replacing into equation (2.3), we find:

$$\alpha = -\frac{1}{2}, \quad \beta = -\frac{1}{4}.$$

Thus, the general solution is:

$$y_G = -\frac{1}{4}(2t+1)e^t + \lambda e^{-t} + \mu e^{2t}, \quad \lambda, \mu \in \mathbb{R}.$$

Example 2.4.3.

Consider the equation:

$$y'' - y' - 2y = \sin(2t)$$

The general solution of the homogeneous equation is:

$$y_H = \lambda e^{-t} + \mu e^{2t}, \quad \lambda, \mu \in \mathbb{R}.$$

We look for a particular solution of the form:

$$y_p = \alpha \cos(2t) + \beta \sin(2t).$$

We have:

$$y_p' = -2\alpha \sin(2t) + 2\beta \cos(2t),$$

$$y_p'' = -4\alpha \cos(2t) - 4\beta \sin(2t).$$

Replacing in the equation, we get:

$$-2(3\alpha + \beta)\cos(2t) + (2\alpha - 6\beta)\sin(2t) = \sin(2t)$$

so that

$$\begin{cases} 3\alpha + \beta = 0 \\ 2\alpha - 6\beta = 1 \end{cases} \implies \alpha = \frac{1}{20}, \quad \beta = -\frac{3}{20}.$$

Thus, the general solution is:

$$y_G = \frac{1}{20}\cos(2t) - \frac{3}{20}\sin(2t) + \lambda e^{-t} + \mu e^{2t}, \quad \lambda, \mu \in \mathbb{R}.$$

2.5 Homogeneous linear equation with analytic coefficients

Consider the second-order homogeneous differential equation:

$$y'' + a(t)y' + b(t)y = 0. \tag{2.4}$$

Suppose that $a(t)$ and $b(t)$ are given by power series (i.e., series in non-negative integer powers of t):

$$a(t) = \sum_{k=0}^{+\infty} a_k t^k, \qquad b(t) = \sum_{k=0}^{+\infty} b_k t^k.$$

We look for a solution of equation (2.4) in the form of a power series:

$$y(t) = \sum_{k=0}^{+\infty} c_k t^k = c_0 + c_1 t + c_2 t^2 + \dots$$

Example 2.5.1.

Consider the second-order equation:

$$y'' - ty' - 2y = 0. \tag{2.5}$$

We seek a solution as a power series:

$$y_1(t) = \sum_{k=0}^{+\infty} c_k t^k,$$

then

$$y_1'(t) = \sum_{k=1}^{+\infty} k c_k t^{k-1}, \qquad y_1''(t) = \sum_{k=2}^{+\infty} k(k-1) c_k t^{k-2}.$$

Substituting y_1, y_1', and y_1'' into equation (2.5) gives:

$$\sum_{k=2}^{+\infty} k(k-1) c_k t^{k-2} - t \sum_{k=1}^{+\infty} k c_k t^{k-1} - 2 \sum_{k=0}^{+\infty} c_k t^k = 0$$

$$\Longrightarrow \sum_{k=2}^{+\infty} k(k-1) c_k t^{k-2} - \sum_{k=1}^{+\infty} k c_k t^k - 2 \sum_{k=0}^{+\infty} c_k t^k = 0.$$

By changing the index in the first sum ($p = k - 2 \Longrightarrow k = p + 2$), we obtain:

$$2c_2 + \sum_{p=1}^{+\infty} [(p+2)(p+1) c_{p+2} - (p+2) c_p] t^p - 2c_0 - 2 \sum_{p=1}^{+\infty} c_p t^p = 0$$

$$\Longrightarrow 2c_2 - 2c_0 + \sum_{p=1}^{+\infty} [(p+2)(p+1) c_{p+2} - (p+3) c_p] t^p = 0.$$

Setting the coefficients of all powers of t to zero allows us to recursively determine

$c_0, c_1, \dots.$

With initial conditions $y_1(0) = 1$ and $y_1'(0) = 0$, we find $c_0 = 1$ and $c_1 = 0$. Then we recursively obtain:

$$c_2 = 0, \quad c_3 = \frac{1}{2}c_1 = 0, \quad c_4 = \frac{1}{3}c_2 = \frac{1}{3}, \dots$$

Thus:

$$y_1(t) = 1 + t^2 + \frac{1}{3}t^4 + \dots$$

Similarly, taking a second solution of the form $y_2(t) = \sum_{k=0}^{+\infty} \alpha_k t^k$ with initial conditions

$$y_2(0) = 0, \quad y_2'(0) = 1,$$

we get $\alpha_0 = 0, \alpha_1 = 1$, and by substitution in (2.5):

$$\alpha_{k+2} = \frac{1}{k+1}\alpha_k, \quad k = 0, 1, 2, \dots$$

$$\Longrightarrow \alpha_{2k} = 0, \quad \alpha_{2k+1} = \frac{1}{k!2^k}.$$

Hence:

$$y_2(t) = \sum_{k=0}^{+\infty} \frac{1}{k!2^k} t^{2k+1} = t \sum_{k=0}^{+\infty} \frac{1}{k!} \left(\frac{t^2}{2}\right)^k = te^{t^2/2}.$$

The general solution of (2.5) is:

$$y(t) = \lambda y_1(t) + \mu y_2(t), \quad \lambda, \mu \in \mathbb{R}.$$

2.6 Equation of Order n and Differential Systems

Recall that an n-th order differential equation involves an unknown function $y(t)$, its derivatives up to order n, and the variable t, in the form:

$$y^{(n)} = F(t, y, y', \dots, y^{(n-1)}).$$

2.6.1 First-Order Differential System

Definition 2.3 (Canonical System).

A system of ordinary differential equations:

$$F_k(t, y_1, y_1', \dots, y_1^{(k_1)}, y_2, y_2', \dots, y_2^{(k_2)}, \dots, y_n, y_n', \dots, y_n^{(k_n)}) = 0, \quad k = 1, 2, \dots, n, \tag{2.6}$$

is called a **canonical system**.

The **order** of the system (2.6) is the number

$$p = k_1 + k_2 + \cdots + k_n.$$

Integrating this system means determining the functions $y_1, y_2, \dots, y_n$ that satisfy the equations of the system (2.6).

Definition 2.4 (Normal System).

Let $y_1, y_2, \dots, y_n$ be n differentiable functions of the variable t.

A **first-order differential system** is any system of differential equations of the form:

$$\begin{cases} \dfrac{dy_1}{dt} = F_1(t, y_1, \dots, y_n) \\ \dfrac{dy_2}{dt} = F_2(t, y_1, \dots, y_n) \\ \vdots \\ \dfrac{dy_n}{dt} = F_n(t, y_1, \dots, y_n) \end{cases}$$

Such a system can be written in vector form as:

$$y_i' = F_i(t, y), \quad i = 1, \dots, n \iff y' = F(t, y), \quad y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}, \quad y' = \begin{pmatrix} y_1' \\ \vdots \\ y_n' \end{pmatrix}, \quad F = \begin{pmatrix} F_1 \\ \vdots \\ F_n \end{pmatrix},$$

where F is continuous on a domain $D \subset \mathbb{R} \times \mathbb{R}^n$.

Theorem 2.6.1 (Existence and Uniqueness).

Let the system $y' = F(t, y)$, where F is continuous on a domain $D \subset \mathbb{R} \times \mathbb{R}^n$ and has continuous partial derivatives with respect to y_i, $i = 1, \dots, n$, on D.

Then, for any $(t_0, y_0) \in D$, there exists a unique maximal solution $u(t)$ satisfying the initial condition

$$u(t_0) = y_0.$$

2.6.2 The solution of a system of n first-order equations

A **solution** of the system (S) is a set of real differentiable functions $y_1(t), \ldots, y_n(t)$ defined on the same interval $I \subset \mathbb{R}$ and satisfying for all $t \in I$:

$$y_i' = F_i(t, y_1(t), y_2(t), \ldots, y_n(t)), \quad i = 1, \ldots, n.$$

Example 2.6.1.

Let a and b be real numbers.

The functions

$$\begin{cases} y_1(t) = a\cos t + b\sin t \\ y_2(t) = -a\sin t + b\cos t \end{cases}, \quad t \in \mathbb{R}$$

are solutions of the system:

$$\begin{cases} y_1' = y_2 \\ y_2' = -y_1 \end{cases}$$

since it is easy to verify that:

$$\begin{cases} y_1' = -a\sin t + b\cos t = y_2 \\ y_2' = -a\cos t - b\sin t = -y_1 \end{cases}$$

2.6.3 The Relationship Between an n-th Order Differential Equation and an n-th Order System

Let an n-th order differential equation be:

$$y^{(n)} = F(t, y, y', \ldots, y^{(n-1)}).$$

By considering the successive derivatives $y, y', \ldots, y^{(n-1)}$ as new unknown functions, we can rewrite it as a differential system:

$$\begin{cases} y_1(t) = y(t) \\ y_2(t) = y'(t) \\ \vdots \\ y_n(t) = y^{(n-1)}(t) \end{cases} \Longrightarrow \begin{cases} y_1'(t) = y'(t) = y_2(t) \\ y_2'(t) = y''(t) = y_3(t) \\ \vdots \\ y_n'(t) = y^{(n)}(t) = F(t, y, y', \dots, y^{(n-1)}) \end{cases}$$

Example 2.6.2.

Consider the third-order equation:

$$y''' = 3y'' - y' + y.$$

It can be rewritten as the differential system:

$$y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{pmatrix} = \begin{pmatrix} y(t) \\ y'(t) \\ y''(t) \end{pmatrix} \Longrightarrow y'(t) = \begin{pmatrix} y_1'(t) \\ y_2'(t) \\ y_3'(t) \end{pmatrix} = \begin{pmatrix} y'(t) \\ y''(t) \\ y'''(t) \end{pmatrix}$$

$$y'(t) = \begin{pmatrix} y_2(t) \\ y_3(t) \\ 3y_3(t) - y_2(t) + y_1(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 3 \end{pmatrix} y(t)$$

2.6.4 Solved Exercises

Solve the following differential equations:

(a) $y'' - 4y' + 3y = (2x+1)e^{-x}$

Solution: The associated homogeneous equation is:

$$y_h'' - 4y_h' + 3y_h = 0$$

with characteristic equation:

$$r^2 - 4r + 3 = 0 \implies r = 1, 3$$

so

$$y_h = C_1 e^{x} + C_2 e^{3x}.$$