Chapter 1

First-order equation

1.1 Fundamental Results

1.1.1 First-order ODE (Ordinary Differential Equation).

Definition 1.1.

A differential equation is a relation of the form

$$F(t, y, y', ..., y^{(n)}) = 0 \tag{1.1}$$

between a variable t, a function y, and its successive derivatives up to order n. [17]

Example 1.1.1.

(1) $yy'' = t - t^2y'$.

(2) $x^{'} - tx + t^2 = 1$.

Definition 1.2.

The order of a differential equation is defined as the order of its highest derivative.[18]

Example 1.1.2.

(1) $yy'' = t - t^2y'$ is a second-order differential equation.

(2) $x^{'} - tx + t^2 = 1$is a first-order differential equation.

Definition 1.3.

A solution (or integral) of a differential equation is any function y = y(t) defined on an interval $\mathrm{I} \subset \mathbb{R}$, possessing successive derivatives $y', y'', \ldots, y^{(n)}$ and satisfying relation (1.1).[17]

Example 1.1.3.

For the differential equation $y''+y=0$ (a second-order differential equation). The function $y_1 = \sin x$ is a solution, since $y_1 = \sin x$ is defined on $I = \mathbb{R}$, and $y_1' = \cos x$, $y_1'' = -\sin x$, hence $y''+y=0$. Similarly, $y_2 = \cos x$ is also a solution of the given equation, and likewise $y_3 = \sin x+\cos x$. In general, $y = a\sin x+b\cos x$ is also a solution, where a and b are two real constants. This last one is called the *general solution*, while the others $y_1, y_2, \ldots, y_n$ are called *particular solutions.*

Remark 1.1.1.

1. To solve or integrate a differential equation means to find all the solutions of this differential equation.
2. The graph of the general solution **(resp. particular solution)** is called the integral curve.

1.1.2 First-order differential equation

Definition 1.4.

A first-order differential equation is any relation of the form:

$$y' = F(t,y) \tag{1.2}$$

1.1.3 Separable differential equations

Definition 1.5.

A separable differential equation is any equation of the form:[17]

$$f(y)y' = g(t) \tag{1.3}$$

where f and g are two numerical functions defined and continuous on intervals to be specified.

Remark 1.1.2.

Equation (1.3) can also be written in the form:

$$f(y)dy = g(t)dt.$$

The solutions of equation (1.3) are defined by

$$\int f(y)dy = \int g(t)dt + c.$$

Proof 1.1.1.

By substituting $Y' = \dfrac{dy}{dt}$ into (1.3), we obtain $f(y)\dfrac{dy}{dt} = g(t)$ or $f(y)\,dy = g(t)\,dt$.

$$f(y)dy = g(t)dt \tag{1.4}$$

We integrate (1.4) term by term in order to obtain the general solution of equation (1.3) in the form: ...

$$\int f(y)dy = \int g(x)dx + c.$$

where **c** is an arbitrary constant.

Example 1.1.4.

Solve on $\mathbb{R}_+^*$ the equation:

$$t^2y' - y^2 = 0$$

It is obvious that $y = 0$ is a solution.

For $y \neq 0$, we can separate the variables since $y' = \dfrac{dy}{dt}$, we obtain:

$$\begin{aligned} t^2y' - y^2 = 0 \Longrightarrow & t^2\frac{dy}{dt} = y^2 \frac{dy}{y^2} = \frac{dt}{t^2} \\ & \Longleftrightarrow \frac{-1}{y} = \frac{-1}{t} + c, \quad c \in \mathbb{R} \\ & \Longleftrightarrow \frac{1}{y} = \frac{1-ct}{t} \\ & \Longleftrightarrow y = \frac{t}{1-ct}, \quad c \in \mathbb{R}. \end{aligned}$$

Therefore, the set S of solutions is

$$S = \{y = 0 \text{ or } y = \frac{t}{1-ct}, c \in \mathbb{R}\}$$

Example 1.1.5.

Integrate the following equation

$$t^3y' = e^{3y}$$

We can separate the variables t and y; we obtain

$$e^{-3y}dy = \frac{dt}{t^3}.$$

By integrating, we obtain

$$\frac{e^{-3y}}{3} = \frac{1}{2t^2} + c,$$

$$\begin{aligned} e^{-3y} = 3\Big(\frac{1}{2t^2} + c\Big) &\Longrightarrow -3y = ln\Big|3\Big(\frac{1}{2t^2} + c\Big)\Big| \\ &\Longrightarrow y = \frac{-1}{3}ln\Big|\frac{3}{2t^2} + \tilde{c}\Big|, \quad \tilde{c} \in \mathbb{R}. \end{aligned}$$

1.1.4 Homogeneous differential equations

Recall :

f is said to be homogeneous of degree n if it satisfies the identity

$$\forall (x, y) \in D_f, f(\lambda x, \lambda y) = \lambda^n = f(x, y).$$

Example 1.1.6.

We show that $f(x, y) = x^2 + y^2 - xy$ is a homogeneous function :

Indeed, for all $(x, y) \in \mathbb{R}^2$ and $\lambda \in \mathbb{R}$, we have :

$$\begin{aligned} f(\lambda x, \lambda y) &= (\lambda x)^2 + (\lambda y)^2 - (\lambda x)(\lambda y) \\ &= \lambda^2 x^2 + \lambda^2 y^2 - \lambda^2 xy \\ f(\lambda x, \lambda y) &= \lambda^2(x^2 + y^2 - xy) \end{aligned}$$

Therefore, f is a homogeneous function of order 2.

Definition 1.6.

A differential equation of the form $y' = f(t, y)$ is called homogeneous if the function $f(t, y)$ is homogeneous of degree zero.[17]

Remark 1.1.3.

A homogeneous differential equation can always be written in the form $y' = \varphi(\frac{y}{t})$.

1.1.5 The solution of a homogeneous differential equation

Consider the homogeneous differential equation

$$y' = \varphi(\frac{y}{t}) \tag{1.5}$$

Let $\frac{y}{t} = u$. It follows that $y = ut \iff y' = u't + u$. Therefore

$$\begin{aligned} y' = \varphi(\frac{y}{t}) \qquad & \iff u't + u = f(u) \\ & \iff tu' + u = f(u) \\ & \iff tu' = f(u) - u \iff \frac{u'}{f(u) - u} = \frac{1}{t} \\ & \iff \frac{du}{f(u) - u} = \frac{dt}{t} \\ & \iff \int \frac{du}{f(u) - u} = \ln|t| + c, \qquad c \in \mathbb{R}. \end{aligned}$$

Therefore, the solutions of equation (1.5) are given by

$$y = tu \quad et \quad t = ke^{\int \frac{du}{f(u) - u}}, \qquad k \in \mathbb{R}.$$

Example 1.1.7.

Solve the following equation

$$ty' = \sqrt{t^2 - y^2} + y \tag{1.6}$$

For $t \neq 0$, it follows that

$$\begin{aligned} (1.6) \iff y' &= \sqrt{\frac{t^2 - y^2}{t^2}} + \frac{y}{t} \\ \iff y' &= \sqrt{1 - (\frac{y}{t})^2} + \frac{y}{t}. \end{aligned} \tag{1.7}$$

Let $\frac{y}{t} = u$, thus $y' = tu' + u$. Then:

$$\begin{aligned}
(1.6) \Longleftrightarrow tu' + u &= \sqrt{1-u^2} + u \\
\Longleftrightarrow tu' &= \sqrt{1-u^2} \\
\Longleftrightarrow \frac{u'}{\sqrt{1-u^2}} &= \frac{1}{t} \Longleftrightarrow \frac{du}{\sqrt{1-u^2}} = \frac{dt}{t} \\
\Longleftrightarrow \arcsin u &= \ln|t| + c, \qquad c \in \mathbb{R} \\
\Longleftrightarrow u &= \sin(\ln|t| + c) \\
\Longleftrightarrow y &= t\sin(\ln|t| + c), \qquad c \in \mathbb{R}
\end{aligned}$$

1.1.6 First-order linear differential equations

Definition 1.7.

The general form of a first-order linear equation is

$$a(t)y' + b(t)y = f(t) \tag{1.8}$$

where a, b, and f are continuous functions on an interval I $\subset \mathbb{R}$.

1. It is linear because the operator $L(y) = a(t)y' + b(t)y$ is linear.
2. If the function $f \equiv 0$ on I, the operator (2.5) is said to be homogeneous, or without a nonhomogeneous term.
3. The general solution is given by $y = y_h + y_p$, where y_h is the general solution of the homogeneous equation, and y_p is a particular solution of equation (2.5).[17]

1.1.7 Method of solution

Let y be the general solution and y_p a particular solution of the equation (2.5).

$$\begin{cases} ay' + by = f(t) \\ ay_p' + by_p = f(t) \end{cases} \tag{1.9}$$

Hence, by difference

$$a(y' - y_p') + b(y - y_p) = 0$$

ou

$$a(y-y_p)'+b(y-y_p)=0$$

Then, $(y-y_p)$ represents the general solution, denoted y_h, of the homogeneous equation (without the nonhomogeneous term). Therefore,

$$y=y_h+y_p$$

Let us now look for the general solution of the homogeneous equation associated with (2.5).

If $a(t)\neq 0$ on I, then:

$$\begin{aligned} a(t)y'+b(t)y=0 &\Longleftrightarrow \frac{y'}{y}=-\frac{b(t)}{a(t)}\\ &\Longleftrightarrow \ln|y|=-\int\frac{b(t)}{a(t)}dt\\ &\Longleftrightarrow y=ke^{-\int\frac{b(t)}{a(t)}}dt, \qquad k\in\mathbb{R}. \end{aligned}$$

Therefore, the general solution of the homogeneous equation is:

$$y_h=ke^{-\int\frac{b(t)}{a(t)}dt}, \qquad k\in\mathbb{R}.$$

It remains to determine y_p, a particular solution of equation (2.5). For this, we can use the method of variation of constants, which consists in seeking a particular solution in the form:

$$y_p=k(t)e^{-\int\frac{b(t)}{a(t)}dt},$$

Let us suppose that the constant k is a function of t. Then we determine $k(t)$ from the complete equation (2.5).

Example 1.1.8.

Integrate the following differential equation:

$$ty'+t+y=0 \qquad pour \qquad t\neq 0, \tag{1.10}$$

We now solve the homogeneous equation associated with (1.10):

$$\begin{aligned} ty' + y = 0 &\Longleftrightarrow ty' = -y \\ &\Longleftrightarrow y = 0 \quad ou \quad \frac{y'}{y} = \frac{-1}{t} \\ &\Longleftrightarrow y = 0 \quad ou \quad \frac{dy}{y} = -\frac{dt}{t} \\ &\Longleftrightarrow y = 0 \quad ou \quad (\ln|y| = -\ln|t| + c, \qquad c \in \mathbb{R}) \\ &\Longleftrightarrow y = 0 \quad ou \quad y = \frac{k}{t}, \quad k \in \mathbb{R} \\ &\Longleftrightarrow y(t) = \frac{k}{t}, \quad k \in \mathbb{R}. \end{aligned}$$

Therefore, the general solution of the homogeneous equation is

$$y_h(t) = \frac{k}{t}, \qquad k \in \mathbb{R}.$$

We look for a particular solution of the form

$$y_p(t) = \frac{k(t)}{t}.$$

$$\begin{aligned} (1.10) &\Longleftrightarrow ty_p' + y_p = -t \\ &\Longleftrightarrow t\Big(\frac{tk'(t) - k(t)}{t^2}\Big) + \frac{k(t)}{t} = -t \\ &\Longleftrightarrow k'(t) = -t \\ &\Longleftrightarrow k(t) = -\frac{t^2}{2}. \end{aligned}$$

Thus, we can take

$$y_p(t) = -\frac{t}{2}$$

as a particular solution, and consequently the general solution of (1.10) is given by:

$$y(t) = \frac{k}{t} - \frac{t}{2}, \qquad k \in \mathbb{R}$$

$$y(t) = y_h(t) + y_p(t) = \frac{k}{t} - \frac{t}{2}, \quad k \in \mathbb{R}.$$

1.1.8 Nonlinear differential equations

1.1.9 Bernoulli differential equation

Definition 1.8.

We call a **Bernoulli equation** a first-order differential equation that can be written in the form:

$$y'(t) + a(t)y(t) = b(t)\, y^{\alpha}(t),$$

where $a(t)$ and $b(t)$ are continuous functions on an interval I, and $\alpha \in \mathbb{R}$.

$$a(t)y' + b(t)y = f(t)y^{\alpha}, \qquad \alpha \in \mathbb{R}^{*} - \{1\},$$

where a, b, and f are continuous functions on an interval I, and $a(t) \neq 0$ on I.

- **Note:** For $\alpha = 0$ and $\alpha = 1$, the equation reduces to a linear differential equation.[17]

1.1.10 Method of solution

By dividing by y^{α}, we obtain:

$$a(t)y'y^{-\alpha} + b(t)y^{1-\alpha} = f(t)$$

The change of function defined by $z = y^{1-\alpha}$ leads to a linear equation.
Indeed, $z' = (1-\alpha)y'y^{-\alpha}$
therefore $\dfrac{a(t)}{(1-\alpha)}z'(t) + b(t)z(t) = f(t)$
or again

$$a(t)Z' + (1-\alpha)b(t)Z = (1-\alpha)f(t).$$

Example 1.1.9.

Integrate the equation $(y \neq 0)$

$$y' - ty = ty^{3} \tag{1.11}$$

Dividing (2.6) by y^{3}, we obtain:

$$\frac{y'}{y^{3}} - t\frac{1}{y^{2}} = t,$$

then we make the change of variable

$$z = \frac{1}{y^2} = y^{-2},$$

which gives

$$z' = -2y'y^{-3}.$$

By substituting into (2.6), we obtain a first–order linear differential equation.

$$\frac{y'}{y^3} - t\frac{1}{y^2} = t$$
$$-2y'y^{-3} + 2t\frac{1}{y^2} = 2t$$

$$z' + 2tz = 2t \tag{1.12}$$

Solving (1.12) yields:

$$z_H = ke^{-t^2}, \quad z_P = 1 \Longrightarrow z = 1 + ke^{-t^2}, \qquad k \in \mathbb{R}.$$

Therefore, the general solution of (2.6) is:

$$y^2 = \frac{1}{z} \Longrightarrow y = \pm\sqrt{\frac{1}{z}} \quad \text{where} \quad y = \pm\sqrt{\frac{1}{1+ce^{-t^2}}}.$$

1.1.11 Riccati differential equation

Definition 1.9.

A Riccati equation is called a first–order differential equation that can be written in the form:

$$y' = a(t)y^2 + b(t)y + c(t)$$

where a, b and c are continuous functions on an interval I, and $a(t) \neq 0$ on the interval I.[17]

Solution Method

- **Method 01:** (Transforming the Riccati equation into a Bernoulli equation)

Let the Riccati equation be:

$$a(t)y' + b(t)y + c(t)y^2 = f(t) \tag{1.13}$$

Let y_p be a particular solution of equation (1.13). Set $y = y_p + z$, hence $y' = y_p' + z'$. Thus, determining y reduces to determining z. Indeed:

$$\begin{aligned}
(1.13) &\Longleftrightarrow a(y_p' + z') + b(y_p + z) + c(y_p + z)^2 = f(t)\\
&\Longleftrightarrow ay_p' + az' + by_p + bz + c(y_p^2 + 2y_pz + z^2) = f(t)\\
&\Longleftrightarrow \underbrace{ay_p' + by_p + cy_p^2}_{\text{=f(t)}} + az' + (b + 2cy_p)z + cz^2 = f(t)\\
&\Longleftrightarrow az' + (b + 2cy_p)z + cz^2 = f(t).
\end{aligned}$$

By solving this last equation (a Bernoulli equation), we obtain z and consequently the solution y of the Riccati equation (1.13).

- **Method 02:** (Transforming the Riccati equation into a linear equation)

Let the Riccati equation be:

$$a(t)y' + b(t)y + c(t)y^2 = f(t)$$

Let y_p be a particular solution of equation (1.13). Set $y = y_p + \frac{1}{z}$, hence $y' = y_p' - \frac{z'}{z^2}$. Therefore, the determination of y reduces to determining z, indeed:

$$\begin{aligned}
(1.13) &\Longleftrightarrow a\Big(y_p' - \frac{z'}{z^2}\Big) + b\Big(y_p + \frac{1}{z}\Big) + c\Big(y_p^2 + \frac{1}{z}\Big)^2 = f(t)\\
&\Longleftrightarrow ay_p' - a\frac{z'}{z^2} + by_p + \frac{b}{z} + c\Big(y_p^2 + 2\frac{y_p}{z} + \frac{1}{z^2}\Big) = f(t)\\
&\Longleftrightarrow \underbrace{ay_p' + by_p + cy_p^2}_{\text{=f(t)}} - a\frac{z'}{z^2} + \frac{b}{z} + 2c\frac{y_p}{z} + \frac{c}{z^2} = f(t)\\
&\Longleftrightarrow -a\frac{z'}{z^2} + (b + 2cy_p) + \frac{c}{z^2} = 0\\
&\Longleftrightarrow az' - (b + 2cy_p) - c = 0.
\end{aligned}$$

By solving this last equation (first order), we obtain z and consequently the solution y of the Riccati equation (1.13).

Example 1.1.10.

Intégrer l'équation :

$$y' = y^2 - 2ty + t^2 + 1 \tag{1.14}$$

It is easy to verify that $y_p = t$ is a particular solution of (1.14).

By setting $y = t + z$, we have:

$$\begin{aligned} y' = 1 + z' &\Longleftrightarrow 1 + z' = (t+z)^2 - 2t(t+z) + t^2 + 1 \\ &\Longleftrightarrow 1 + z' = 1 + z^2 \\ &\Longleftrightarrow z' = z^2 \Longrightarrow \int \frac{dz}{z^2} = \int dt \Longrightarrow \frac{-1}{z} = t + c \\ &\Longleftrightarrow \frac{1}{z} = -(t+c) \Longrightarrow z = \frac{-1}{t+c} \Longrightarrow y = \frac{-1}{t+c} + t, \end{aligned}$$

Exercice 1.1.1. *1. Solve the following first-order differential equations:*

- *(a)* $y^{'} = 3y$
- *(b)* $y^{'} = 2\frac{y}{x} - 1$
- *(c)* $y^{'} - ty = t$
- *(d)* $y^{'} + \frac{1}{\sqrt{t}}y = \frac{1}{\sqrt{t}}, t \succ 0$
- *(e)* $(t^2 - 1)y^{'} - y = t^2, t \in]1, +\infty[$

Exercice 1.1.2. *1. Solve the following first-order differential equations:*

- *(a)* $\frac{dy}{dt} + \frac{y}{t} = 4$
- *(b)* $y^{'} \cos t + y \sin t = 1$
- *(c)* $\frac{dy}{dt} = (1 + y^2)e^t$
- *(d)* $dy + y \tan t = 0$

Exercice 1.1.3. *1. Solve the following first-order differential equations:*

- *(a)* $y^{'} + y - 5e^{-t}y^5 = 4$
- *(b)* $y^{'} - y = ty^5$
- *(c)* $y^{'} + p(t)y + q(t)y^r = 0, r \in \mathbb{R}$

Exercice 1.1.4. *1. Solve the following first-order differential equations: .*

- *(a)* $y^{'} - \frac{1}{t}y - y^2 = -9t^2, y_p = at, t \in]0, +\infty[$

(b) $t^3y^{'} + y^2 + t^2y + 2t^4 = 0, y_p = -t^2.$

1.2 Existence and Uniqueness of Solutions

1.2.1 Maximal and global solutions

Let the O.D.E.

$$\frac{dy}{dt} = f(t, y),$$

where f is defined on the open set U of $I \times \mathbb{R}^n$; $\quad U = I \times \mathbb{R}^n$; $\quad f = I \times \Omega \longrightarrow \mathbb{R}$.

Extension of a solution

[5] Let y be a solution of (E) on $J \subset I$.

Let $\tilde{y}$ be a solution of (E) on $\tilde{J} \subset I$.

We say that $\tilde{y}$ is an **extension** of y if :

$$J \subset \tilde{J} \quad \text{and} \quad \tilde{y}_{|J} = y.$$

$\alpha)$ $J \subset \tilde{J}$.

$\beta)$ $\tilde{y}_{/J} = y \quad (\tilde{y}(t) = y(t), \quad \forall t \in J), \quad \tilde{y} = y$ on J.

Example 1.2.1.

Let us consider on $I = (0, +\infty)$ **the equation**

$$y'(t) = \frac{2}{t}y(t) \Longleftrightarrow \frac{dy}{dt} = \frac{2}{t}y \Longrightarrow \frac{dy}{y} = \frac{2}{t} \Longrightarrow \ln|y| = 2\ln t + c$$

$$\Longrightarrow |y| = kt^2 \Longrightarrow y = \pm kt^2 \Longrightarrow y = ct^2.$$

Let the solution be $y :]3, +\infty[\subset I \longmapsto \mathbb{R}$.

A solution defined by $y(t) = t^2$.

The solution $\tilde{y} :]2, +\infty[\longmapsto \mathbb{R}$ defined by :

$\tilde{y}(t) = t^2$ is an extension of y because :

- $J \subset \tilde{J} \quad (]3, +\infty[\subset]2, +\infty[)$. • $y(t) = \tilde{y}(t), \forall t \in J =]3, +\infty[$.

1.2.2 Maximal and global solutions

[5] A solution $y : J \subset I \longmapsto \mathbb{R}$ is said to be **maximal** if it does not admit **any extension** $\tilde{y} : \tilde{J} \longmapsto \mathbb{R}$ with $J \subseteq \tilde{J} \quad$ (J strictly included in $\tilde{J}$).

Example 1.2.2.

The function defined on $J = \mathbb{R}$ by $y(t) = e^{-4t}$ is a maximal solution of the differential equation $y' = -4y(t)$, since it is defined on $\mathbb{R}$ which is a maximal interval.

Theorem 1.2.1.

Let $\alpha, \beta \in \mathbb{R}$.
Let y be a solution of (E) defined on $]\alpha, +\infty[\quad (]-\infty, \beta[)$ if :

$$\lim_{t \overset{>}{\to} \alpha} y(t) \quad (\lim_{t \overset{<}{\to} \beta} y(t)) \quad doesnot \quad exist.$$

Then y is a **maximal** solution

Example 1.2.3.

The function $y : J =]-\infty, -1[\to \mathbb{R}$ defined by :

$$y(t) = \frac{1}{1+t}$$

is a maximal solution of $y'(t) = -y^2(t)$ because:

$$\lim_{t \overset{<}{\to} -1} \frac{1}{1+t} = -\infty.$$

Remark 1.2.1.

If y is a maximal solution on the interval I, this does not imply that y is the unique solution of (E) on I.

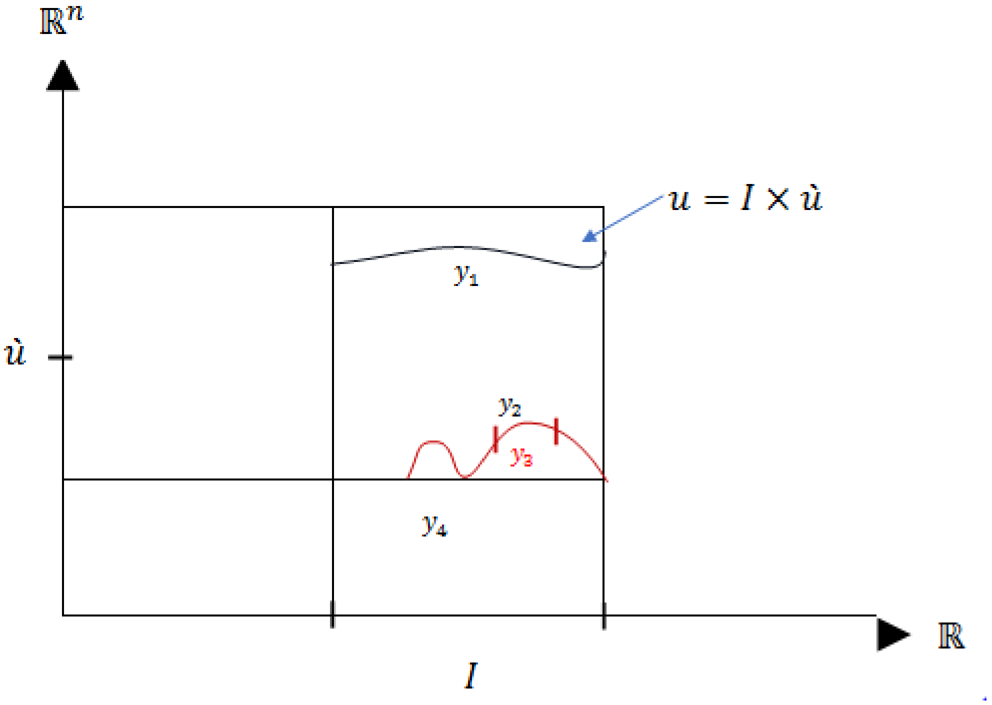
1.2.3 The global solution

[5] Let the ODE:

$$\frac{dy}{dt} = f(t, y),$$

where f is defined on the open set $U \subset \mathbb{R} \times \mathbb{R}^n$ such that $U = I \times \Omega$.

The function y is called a *global solution* if it is defined on the entire interval I.



- y_1 : **Global maximal solution**.
- y_3 : **Neither maximal nor global solution**
- y_2 :**Neither maximal nor global solution** • y_4 :**Maximal but not global solution**.

Remark 1.2.2.

Every global solution is maximal, but the converse is false.

Example 1.2.4.

We have: $\frac{dy}{dt} = y^2$ and $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ où $f(t, y) = y^2$ and $\begin{cases} I = \mathbb{R} \\ \Omega = \mathbb{R} \end{cases}$ The solutions are :

- $y_1(t) = 0, \quad \forall t \in \mathbb{R}$ **Global solution: it is defined** I.
- $y_2(t) = -\frac{1}{t}, \quad t \in]-\infty, 0[$. • $y_3(t) = -\frac{1}{t}, \quad t \in]0, +\infty[$.

y_2 et y_3 are maximal solutions.

Remark 1.2.3.

Every solution of (E) can be extended to a maximal solution.

1.2.4 Local and global existence, uniqueness

[5] A function of several variables is said to be of class C^k if it admits continuous partial derivatives up to order k.

- **For** $k = 1$**:**

Let f be a function defined on an open subset D of $\mathbb{R}^2$ with values in $\mathbb{R}$.
The function f is of class C^1 on D if $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ exist and are continuous on D.

- **For** $k = 2$**:**

The function f is of class C^2 on D

$$\Longleftrightarrow \frac{\partial^2 f}{\partial x^2}, \quad \frac{\partial^2 f}{\partial x \partial y}, \quad \frac{\partial^2 f}{\partial y \partial x}, \quad \frac{\partial^2 f}{\partial y^2}$$

exist and are continuous on D.

Example 1.2.5.

We have : $f(x,y) = xe^{xy}$
$\dfrac{df}{dx}(x,y) = e^{xy} + y \cdot xe^{xy}$ continue on $\mathbb{R}^2$.
$\dfrac{df}{dy}(x,y) = x \cdot xe^{xy}$ continue on $\mathbb{R}^2$.

That is to say, f is of class C^2 on $\mathbb{R}^2$.

Let I be an open interval of $\mathbb{R}$, and Ω an open set in $\mathbb{R}^n$. The function f is defined on $I \times \Omega$.

Lemma 1.2.1. ***(Gronwall integral)***
Let φ be a continuous function from $[a, b]$ into $\mathbb{R}_+$ and $c \in [a, b]$. Suppose that there exist positive constants A and B such that:

$$\varphi(t) \leq A + B|\int_c^t \varphi(s)ds|, \qquad t \in [a, b]$$

Then:

$$\varphi(t) \leq Ae^{B|t-c|}, \qquad t \in [a, b].$$

Proof 1.2.1.

Let $t \in [a, b]$, and suppose that $t \geq c$. Define:

$$F(t) = A + B \int_c^t \varphi(s)\, ds.$$

Then:

$$F \in C^1 \text{ and } \varphi(t) \leq F(t) \text{ for } t \in [c, b].$$

We have:

$$\begin{aligned} F'(t) &= B\varphi(t), \\ \frac{d}{dt}\Big(e^{-Bt}F(t)\Big) &= -Be^{-Bt}F(t) + F'(t)e^{-Bt} \\ &= e^{-Bt}\Big(F'(t) - BF(t)\Big) \leq 0 \\ &= e^{-Bt}\Big(B\varphi(t) - BF(t)\Big) \leq 0, \end{aligned}$$

for $t \in [c, b]$.

Thus:

$$\begin{aligned} &\frac{d}{dt}\Big(e^{-Bt}F(t)\Big) \leq 0 \\ &\quad \Rightarrow e^{-Bt}F(t) \leq e^{-Bc}F(c) = Ae^{-Bc} \quad (\text{since } F(c) = A). \end{aligned}$$

Hence:

$$e^{-Bt}\varphi(t) \leq e^{-Bt}F(t) \leq Ae^{-Bc},$$

and therefore:

$$\varphi(t) \leq Ae^{B(t-c)}.$$

In the same way, the result can be shown for $t \leq c$.

1.2.5 Regularity of solutions

Theorem 1.2.2.

[6] If $f : \Omega \subset \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is of class C^k, then every solution of (E) is of class C^{k+1}.

Proof 1.2.2.

We proceed by induction on k.

$$P(n) : f \in C^n \Longrightarrow y \in C^{n+1}$$

By assumption, $y : I \to \mathbb{R}$ is differentiable (y' exists) and continuous (since $f(t, y)$ is continuous).
Thus, y is of class C^1.

- If the result is true for order "$k+1$", then y is at least of class C^k.
- Since f is of class C^k, it follows that $y' = f(t, y)$ is of class C^k (as the composition of C^k functions), hence y is of class C^{k+1}.

Proposition 1.2.1.

1.2.6 Cauchy–Lipschitz theorems

[5]

Definition 1.10.

We say that f is **locally Lipschitz** with respect to its second variable if, for each $(t_0, y_0) \in \underbrace{I \times \Omega}_{\text{U}}$, there exists a neighborhood V of (t_0, y_0) such that the restriction $f_{/V}$ is Lipschitz with respect to the second variable.

Definition 1.11.

We say that f is **globally Lipschitz** with respect to the second variable if there exists $k > 0$ such that:

$$\|f(t, y_1) - f(t, y_2)\| \leq k\|y_1 - y_2\|, \quad \forall (t, y_1, y_2) \in I \times \Omega \times \Omega.$$

Example 1.2.6.

1. $f(t, y) = t^4 + 5y, \quad \forall t \in \mathbb{R}, \quad \forall (y_1, y_2) \in \mathbb{R}^2.$

   $$|f(t, y_1) - f(t, y_2)| = |(t^4 + 5y_1) - (t^4 + 5y_2)| = 5|y_1 - y_2|.$$

   Thus, the function f is globally Lipschitz with respect to the second variable y.

2. $g(t, y) = \cos(y)$. By the Mean Value Theorem, $\forall (y_1, y_2) \in \mathbb{R}^2$, $\forall t \in \mathbb{R}$, there exists $c \in]y_1, y_2[$ such that:

   $$\cos(y_1) - \cos(y_2) = \cos'(c)(y_1 - y_2).$$

   Hence:

   $$\begin{aligned} |\cos(y_1) - \cos(y_2)| &= |-\sin(c)(y_1 - y_2)| \\ &= |\sin(c)|\,|y_1 - y_2| \\ &\leq |y_1 - y_2|. \end{aligned}$$

   Therefore, g is globally Lipschitz with respect to the second variable.

3. $h(t, y) = \sqrt{y}$ is not locally Lipschitz in a neighborhood of 0. Indeed:

   $$\lim_{y \to 0^+} \frac{h(t, y) - h(t, 0)}{y - 0} = \frac{\sqrt{y}}{y} = \frac{1}{\sqrt{y}} \longrightarrow_{y \to 0^+} +\infty.$$

Remark 1.2.4.

(1) If $f \in C^1(U)$, then f is locally Lipschitz.

(2) A Lipschitz mapping is locally Lipschitz and continuous.

1.2.7 Differential equation (Cauchy problem)

Definition 1.12 (Contraction)**.**

A **contraction** f on $A \subset E$ is a mapping

$$f : A \longrightarrow A$$

which is k-Lipschitz with $k \in [0, 1[$.

Remark 1.2.5.

$$f \text{ contraction } \implies f \text{ uniformly continuous } \implies f \text{ continuous.}$$

Theorem 1.2.3 (Banach fixed point)**.**

[11] Every contraction f of a non-empty closed subset of a Banach space has a unique fixed point (i.e., $f(x) = x$).

Proof 1.2.3.

Remark. There is uniqueness of the fixed point in case of existence.

Indeed, if x and y are two fixed points, then

$$\|x - y\| = \|f(x) - f(y)\| \leq k\|x - y\|.$$

Since $k \in [0, 1[$, necessarily $\|x - y\| = 0$, hence $x = y$.

To establish the existence of a fixed point, let us take $x_0 \in A$ and define the sequence recursively by $x_{n+1} = f(x_n)$. We then have:

$$\begin{aligned}
\|x_{n+1} - x_n\| &\leq k\|x_n - x_{n-1}\| \\
&\leq k^2\|x_{n-1} - x_{n-2}\| \\
&\vdots \\
&\leq k^n\|x_1 - x_0\|.
\end{aligned}$$

From the triangle inequality, for $m < n$ we obtain:

$$\begin{aligned}
\|x_n - x_m\| &\leq \|x_n - x_{n-1}\| + \|x_{n-1} - x_{n-2}\| + \cdots + \|x_{m+1} - x_m\| \\
&\leq \left(k^{n-1} + k^{n-2} + \cdots + k^m\right) \|x_1 - x_0\| \\
&\leq \frac{k^m}{1-k} \|x_1 - x_0\|.
\end{aligned}$$

Thus (x_n) is a Cauchy sequence in A. Since A is a closed subset of the Banach space E, there exists $x^* \in A$ such that

$$\lim_{n \to \infty} x_n = x^*.$$

Finally, by continuity of f, we get $f(x^*) = x^*$, which is the unique fixed point.

Theorem 1.2.4.

If $f : \Omega \to \mathbb{R}^n$ is a function of class C^1 on an open set $\Omega \subset \mathbb{R} \times \mathbb{R}^n$, then for each $(t_0, y_0) \in \Omega$, there exists a **unique** solution of the equation $y' = f(t, y)$ with $y(t_0) = y_0$ on some open interval containing t_0.

Theorem 1.2.5.

If the function $f(t, y)$ is continuous in the domain U and has in U a bounded derivative $f_y(t, y)$, then through each point (t_0, y_0) of U passes one and only one integral curve $y = \varphi(t)$ of the equation:

$$y' = f(t, y) \text{ with } (\varphi(t_0) = y_0).$$

1.2.8 Existence and uniqueness of the solution satisfying an initial condition

[5]

Theorem 1.2.6.

If the functions f and $\frac{df}{dy}$ are continuous in Ω and if (t_0, y_0) is a point of Ω, then there exists a unique solution φ, defined in a neighborhood of (t_0, y_0), which satisfies $\varphi(t_0) = y_0$.

Example 1.2.7.

Let the differential equation be:

$$y' = \alpha y \text{ with } \alpha \in \mathbb{R}.$$

We have:

$$\begin{cases} f(t,y) = \alpha y \\ \text{and} \\ \dfrac{\partial f}{\partial y}(t,y) = \alpha \end{cases}$$

The functions f and $\dfrac{\partial f}{\partial y}$ are continuous in the whole plane (t,y). The preceding theorem shows that there exists a unique solution passing through the point (t_0, y_0) of the plane. This solution is $\varphi(t) = y_0\, e^{\alpha(t-t_0)}$, which is the only solution passing through this point and it is defined for all t.

Example 1.2.8.

Let the differential equation be:

$$y' = f(t,y) = \frac{y}{t+1}$$

The functions f and $\dfrac{\partial f}{\partial y}$ are continuous in the whole plane (t,y), except on the line $t = -1$. The preceding theorem shows that there exists a unique solution passing through any point (t_0, y_0) of the plane (t,y), but it provides no information about the solutions passing through $(-1, y_0)$.

1.2.9 Global existence and uniqueness

Consider the following Cauchy problem:[6]

$$\begin{cases} y' = \psi(t,y), \\ y(t_0) = y_0 \end{cases} \qquad (PC)$$

Theorem 1.2.7 (Global existence and uniqueness)**.**

[8] We suppose that $U = E$ and $\psi \in C(I \times U)$ is a function **globally Lipschitz** with respect to y.

Then for every $y_0 \in U$, the Cauchy problem **(PC)** admits a unique solution. Moreover, every local solution is a restriction of this one.

Proof 1.2.4.

First suppose that the interval I is compact. We set

$$\varepsilon = C(I, E)$$

the set of continuous functions from I into E, endowed with the norm

$$\|y\|_{\varepsilon} = \max_{t \in I} e^{-2L|t-t_0|} \, \|y(t)\|_E,$$

where L is the Lipschitz constant of ψ.
It is clear that ε is a complete normed space (Banach space), since I is compact. We define the operator $\chi : \varepsilon \to \varepsilon$ by

$$(\chi y)(t) = y_0 + \int_{t_0}^{t} \psi(s, y(s)) \, ds, \quad t \in I.$$

It is clear that the operator χ maps ε into itself.
Suppose $t \geq t_0$. For all $y_1, y_2 \in \varepsilon$, we have:

$$\begin{aligned}
\|(\chi y_2)(t) - (\chi y_1)(t)\|_E &= \left\| y_0 + \int_{t_0}^{t} \psi(s, y_2(s)) \, ds - y_0 - \int_{t_0}^{t} \psi(s, y_1(s)) \, ds \right\|_E \\
&= \left\| \int_{t_0}^{t} \psi(s, y_2(s)) \, ds - \int_{t_0}^{t} \psi(s, y_1(s)) \, ds \right\|_E \\
&\leq \int_{t_0}^{t} \|\psi(s, y_2(s)) - \psi(s, y_1(s))\|_E \, ds \\
&\leq \int_{t_0}^{t} L \, \|y_2(s) - y_1(s)\|_E \, ds.
\end{aligned}$$

1.2.10 Theorems on global solutions

Theorem 1.2.8.

Let $f : I \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuous function. Suppose there exists a continuous function $k : I \to \mathbb{R}$ such that for every $t \in I$ the map $y \mapsto f(t, y)$ is Lipschitz with constant $k(t)$. Then every maximal solution of the problem

$$y' = f(t, y)$$

is global (i.e. defined on the whole interval I).[8]

Example 1.2.9.

Every maximal solution of:

$$\begin{cases} y' = t\sqrt{t^2+y^2}, \\ y(x_0) = y_0 \end{cases}$$

is global. Indeed:

1. For every $t \in \mathbb{R}$, the function

$$f(t,y) = t\sqrt{t^2+y^2}$$

is continuous.

2. The function

$$y \longmapsto f(t,y) = t\sqrt{t^2+y^2}$$

is Lipschitz with constant $k(t) = |t|$ (continuous on $\mathbb{R}$), because for all $y, z \in \mathbb{R}$, we have:

$$\begin{aligned} |f(t,y) - f(t,z)| &\le |t| \frac{|y^2 - z^2|}{\sqrt{t^2+y^2} + \sqrt{t^2+z^2}} \\ &\le |t| \frac{(|y| + |z|)\,|y-z|}{\sqrt{t^2+y^2} + \sqrt{t^2+z^2}} \\ &\le |t|\,|y-z|. \end{aligned} \tag{1.15}$$

since

$$|t| \frac{|y-z|}{\sqrt{t^2+y^2} + \sqrt{t^2+z^2}} \le 1.$$

From Theorem 1.2.10, every maximal solution of (2.1) is global.

Theorem 1.2.9.

Let $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuous function such that:

$$\|f(x,y)\| \le \alpha(t) + \beta(t)\|y\|,$$

for α, β positive and continuous. Then, the solutions of the problem:

$$\begin{cases} y' = f(t,y), \quad t \in \mathbb{R}, \\ y(t_0) = y_0 \end{cases}$$

are global.

Theorem 1.2.10.

Let $f :]a,b[\times \mathbb{R}^n \to \mathbb{R}^n$ be a continuous and bounded function. Then every solution of:

$$\begin{cases} y' = f(t,y), \quad t \in \mathbb{R}, \\ y(t_0) = y_0 \end{cases}$$

is global.

Remark 1.2.6.

[7] The unique solution of the Cauchy problem can be constructed using the following method of successive approximations:

$$\begin{aligned} y_0(t) &= y_0, \\ y_1(t) &= \int_{t_0}^{t} f(s, y_0(s))\, ds + y_0, \\ y_2(t) &= \int_{t_0}^{t} f(s, y_1(s))\, ds + y_0, \\ &\vdots \\ y_n(t) &= \int_{t_0}^{t} f(s, y_{n-1}(s))\, ds + y_0. \end{aligned}$$

Exercice 1.2.1.

1. Show that the function f defined by $f(t,y) = t^2 + y^2$ is locally Lipschitz with respect to y on $\mathbb{R}$.

Proof. Let $f(t,y) = t^2 + y^2$. Fix an arbitrary point $(t_0, y_0) \in \mathbb{R} \times \mathbb{R}$. Choose $M > 0$ such that $|y_0| \leq M$ and consider the strip

$$S_M = \{(t,y) \in \mathbb{R} \times \mathbb{R} : \ |y| \leq M\}.$$

For any $t \in \mathbb{R}$ and any y, z with $|y| \le M,\ |z| \le M$, we have

$$|f(t,y) - f(t,z)| = |y^2 - z^2| = |y-z|\,|y+z|.$$

Since $|y+z| \le |y| + |z| \le 2M$, it follows that

$$|f(t,y) - f(t,z)| \le 2M\,|y-z|.$$

Thus on the neighbourhood S_M the map $y \mapsto f(t,y)$ is Lipschitz with constant $K = 2M$. Because every point (t_0, y_0) admits such a neighbourhood, f is locally Lipschitz in y on $\mathbb{R}$.

Remark. f is not globally Lipschitz in y on $\mathbb{R}$ since the Lipschitz constant $2M$ increases with M and no single finite constant works for all $y \in \mathbb{R}$. ☐

Exercice 1.2.2.

Show that every maximal solution of

$$\begin{cases} y' = t\sqrt{t^2+y^2}, \\ y(t_0) = y_0, \end{cases}$$

is global.

Proof. We give two short proofs.

(I) By a variable Lipschitz constant. Define $f(t,y) = t\sqrt{t^2+y^2}$. For arbitrary $y, z \in \mathbb{R}$ and fixed t we have

$$\begin{aligned} |f(t,y) - f(t,z)| &= |t| \left| \sqrt{t^2+y^2} - \sqrt{t^2+z^2} \right| \\ &= |t| \frac{|y^2 - z^2|}{\sqrt{t^2+y^2} + \sqrt{t^2+z^2}} \le |t|\,|y-z|. \end{aligned}$$

Hence for each t the map $y \mapsto f(t,y)$ is Lipschitz with constant $k(t) = |t|$, and $k(t)$ is continuous on $\mathbb{R}$. Since f is continuous and the Lipschitz constant depends continuously on t, the standard continuation theorem (Grönwall/uniqueness + continuation) implies that any maximal solution cannot blow up in finite time and therefore is defined for all $t \in \mathbb{R}$. Thus every maximal solution is global.

(II) By a priori estimate (Grönwall argument). Let $y(t)$ be a solution on its maximal interval $I_{\max}$. Then

$$|y'(t)| = |t|\sqrt{t^2 + y(t)^2} \leq |t|(|t| + |y(t)|) = t^2 + |t|\,|y(t)|.$$

For t in any finite subinterval of $I_{\max}$ containing t_0 we get

$$|y(t)| \leq |y_0| + \int_{t_0}^{t} \big(s^2 + |s|\,|y(s)|\big)\,ds.$$

Set $z(t) = |y(t)|$ and $C(t) = |y_0| + \int_{t_0}^{t} s^2\,ds$. Then

$$z(t) \leq C(t) + \int_{t_0}^{t} |s|\,z(s)\,ds.$$

Applying Grönwall's inequality on any finite interval yields a finite bound for $z(t)$ on that interval. Hence the solution cannot blow up in finite time and can be extended past any finite endpoint of $I_{\max}$. Therefore $I_{\max} = \mathbb{R}$ and the maximal solution is global. □

Exercice 1.2.3.

Consider the differential equation

$$(E_3) \qquad y' = (1 + \cos t)\,y - y^3.$$

Let $t_0, y_0 \in \mathbb{R}$. Prove existence and uniqueness of the maximal solution y of (E_3) satisfying $y(t_0) = y_0$, and show that this maximal solution is global.

1. Are the following functions Lipschitz in y?

$$f_1(t,y) = \ln(t^2 + y^2 + 1), \qquad f_2(t,y) = 2\sqrt{y}, \quad y \in [1,\infty).$$

2. Show that the function

$$\varphi(t) = \frac{1}{\sqrt{2(2-t)}}$$

defined on its natural domain is a maximal solution of the equation $y' = y^3$.

Proof. **(1) Lipschitz in y.**

For f_1. Compute the partial derivative with respect to y:

$$\frac{\partial f_1}{\partial y}(t,y) = \frac{2y}{t^2 + y^2 + 1}.$$

For every fixed (t, y) we have

$$\left|\frac{\partial f_1}{\partial y}(t,y)\right| = \frac{2|y|}{t^2 + y^2 + 1}.$$

For fixed t the function $g(y) = \dfrac{2|y|}{t^2 + y^2 + 1}$ attains its maximum at $|y| = \sqrt{t^2+1}$, and the maximal value is

$$\max_y g(y) = \frac{1}{\sqrt{t^2+1}} \le 1.$$

Hence for all $(t, y), (t, z)$ one has

$$|f_1(t,y) - f_1(t,z)| \le K\,|y - z|$$

with a Lipschitz constant $K = \sup_t \frac{1}{\sqrt{t^2+1}} = 1$. Thus f_1 is globally Lipschitz in y on $\mathbb{R} \times \mathbb{R}$ (with Lipschitz constant 1).

For f_2. On the domain $y \in [1, \infty)$ we have

$$\frac{d}{dy}(2\sqrt{y}) = \frac{1}{\sqrt{y}}.$$

On $[1, \infty)$ this derivative is bounded by 1. Therefore for all $y, z \ge 1$,

$$|f_2(t,y) - f_2(t,z)| \le 1 \cdot |y - z|.$$

Hence f_2 is (globally) Lipschitz in y on the domain $y \ge 1$ (with Lipschitz constant 1).

(2) φ is a maximal solution of $y' = y^3$.

First note the natural domain of φ. The formula $\varphi(t) = \dfrac{1}{\sqrt{2(2-t)}}$ requires $2(2-t) > 0$, so the domain is

$$(-\infty, 2).$$

(At $t \to 2^-$ the denominator tends to 0^+ and $\varphi(t) \to +\infty$.)

Verification that φ is a solution. Differentiate:

$$\varphi(t) = \big(2(2-t)\big)^{-1/2} \quad \Longrightarrow \quad \varphi'(t) = \frac{d}{dt}\big(2(2-t)\big)^{-1/2} = (2(2-t))^{-3/2}.$$

But

$$\varphi(t)^3 = \big((2(2-t))^{-1/2}\big)^3 = (2(2-t))^{-3/2}.$$

Thus $\varphi'(t) = \varphi(t)^3$ for all $t \in (-\infty, 2)$, so φ is indeed a solution of $y' = y^3$ on $(-\infty, 2)$.

Maximality. Suppose by contradiction that φ could be extended to a solution $\tilde{\varphi}$ defined on a strictly larger interval $(-\infty, 2+\varepsilon)$ (or even defined at $t = 2$). Because $\varphi(t) \to +\infty$ as $t \to 2^-$, any such extension would have to assign a finite value at $t = 2$. But standard ODE continuation theory (continuation/uniqueness and the blow-up criterion) says that a solution of $y' = y^3$ can be continued past a time t^* if and only if its value remains finite as $t \to t^*$. Here the solution diverges to $+\infty$ when approaching $t = 2$, so no extension through $t = 2$ exists. Hence the interval $(-\infty, 2)$ is the maximal interval of existence for φ.

Therefore φ is a maximal solution of $y' = y^3$, defined on $(-\infty, 2)$. □

1. Study the Lipschitz property near 0 of the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(y) = 3\sqrt{|y|}.$$

2. Let $a \geq 0$. Verify that the function $y : \mathbb{R} \to \mathbb{R}$ defined by

$$y(t) = \begin{cases} \dfrac{9}{4}(t-a)^2, & t > a, \\ 0, & t \leq a, \end{cases}$$

is a solution of the Cauchy problem $y' = 3\sqrt{|y(t)|}$ with $y(0) = 0$.

1.3 Dependence on Initial Conditions

In this section, we study the behavior of solutions when the initial conditions are perturbed. Consider the following Cauchy problem (PC):

$$\begin{cases} \frac{dX}{dt} = f(t, X), \\ X(t_0) = X_0, \end{cases} \qquad (t_0, X_0) \in I \times E.$$

We state the following theorem:

Let $I = [a, b]$ be a bounded interval of $\mathbb{R}$. Let $f : I \times E \to \mathbb{R}^n$ be Lipschitz continuous with respect to the second variable, uniformly on I. Consider the Cauchy problem

$$\begin{cases} \frac{dY}{dt} = f(t, Y), \\ Y(t_0) = Y_0, \end{cases} \qquad Y_0 \in E \subset C(I, \mathbb{R}^n).$$

Define the function $\Phi : E \to C(I, \mathbb{R}^n)$ by

$$\Phi(X)(t) = X(t),$$

where $X(t)$ is the solution of (PC) with initial condition $X_0 = X$. Then Φ is continuous with respect to X on E.

Proof. We show that Φ is continuous with respect to the second variable on E. Let $X, Y \in E$, and let $X(t)$ and $Y(t)$ be the solutions of (PC) with initial conditions X and Y, respectively. We have

$$X(t) = X + \int_{t_0}^{t} f(s, X(s))\, ds, \quad Y(t) = Y + \int_{t_0}^{t} f(s, Y(s))\, ds, \quad t \in I.$$

By definition of Φ, we have $\Phi(X)(t) = X(t)$ and $\Phi(Y)(t) = Y(t)$. Then, for all $t \in I$,

$$\Phi(X)(t) - \Phi(Y)(t) = X - Y + \int_{t_0}^{t} \big(f(s, X(s)) - f(s, Y(s))\big)\, ds.$$

Let $J = [t_0, t]$ if $t \geq t_0$ and $J = [t, t_0]$ if $t < t_0$. Define

$$\|X - Y\| = \max_{s \in I} \|X(s) - Y(s)\|.$$

Since $X(t_0) = X$ and $Y(t_0) = Y$, we get

$$\|\Phi(X)(t) - \Phi(Y)(t)\| \leq \|X - Y\| + \int_J \|f(s, X(s)) - f(s, Y(s))\|\, ds.$$

Using the Lipschitz property of f, we obtain

$$\|\Phi(X)(t) - \Phi(Y)(t)\| \leq \|X - Y\| + \int_J L\|X(s) - Y(s)\|\, ds,$$

where L is the Lipschitz constant. Applying Gronwall's inequality yields

$$\|\Phi(X)(t) - \Phi(Y)(t)\| \leq \|X - Y\| e^{L(b-a)}.$$