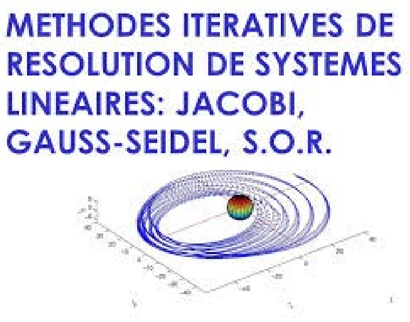
Chapter V:

Chapter 5. Numerical resolution of systems of linear equations

5. 1 Gauss's method

5.2 Gauss-Seidel method



Introduction:

Solving systems of linear equations is a fundamental problem in applied mathematics, physics, and engineering. It occurs in various fields such as fluid mechanics, structural analysis, economics, and computational chemistry.

V.2 Definitions Systems of linear equations :

A square system of linear equations with real coefficients is written in the form of the equations:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots\dots\dots\dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots\dots\dots\dots + a_{2n}x_n = b_2 \\ . \\ . \\ a_{n1}x_1 + a_{n2}x_2 + \dots\dots\dots\dots + a_{nn}x_n = b_n \end{cases}$$

Where in matrix form:

$$AX = B$$

With

$$A = \begin{bmatrix} a_{11} \ a_{12} & & a_{1n} \\ a_{21} \dots\dots\dots\dots a_{2n} \\ . \\ . \\ a_{n1} \dots\dots\dots\dots a_{nn} \end{bmatrix} ; \ X = \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{bmatrix} ; B = \begin{bmatrix} b_1 \\ b_2 \\ . \\ . \\ b_n \end{bmatrix}$$

This system is therefore made up of n equations and n unknowns. The elements (a_{ij}) of the matrix (A) and (bi) of the vector (B) are real data, while those (xi) of the vector (X) are unknown reals. The resolution of the system of linear equations therefore comes down to the determination of the vector X.

In most cases we deal with non- **singular** or **invertible matrices** , i.e. those whose matrix (A^{-1}) exists.

- Definitions :

- A matrix A is said to be non-singular or invertible if it has an inverse, that is to say a matrix A^{-1} such that:

$$A^{-1} \times A = I$$

Where I is the identity matrix

Condition the determinant of A must be different from zero

- Diagonal matrix: A matrix is said to be diagonal if all its elements outside the main diagonal are zero:

$$(a_{ij} = 0 \text{ for } i \neq j)$$

$$A = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$$

- Upper triangular matrix: A matrix is said to be upper triangular if all its elements located below the main diagonal are zero.

$$(a_{ij} = 0 \text{ for } i > j)$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$$

- Lower triangular matrix: A matrix is said to be lower triangular if all its elements located above the main diagonal are zero.

$$(a_{ij} = 0 \text{ for } j > i)$$

$$A = \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

V.3 <u>Different resolution methods</u>

Methods for solving linear systems fall into two broad categories:

(a) . **<u>Direct methods</u>**

These methods provide an exact solution in a finite number of steps (assuming exact calculations).

- **Gaussian Elimination** : Used to reduce the matrix A into a triangular shape, making it easier to solve.
- **LU decomposition** : Factorizes A into a product of two matrices (lower triangular and upper triangular).

⌑ **Cholesky methods** : Used for positive definite symmetric matrices.

(b). **Iterative methods**

These methods start with an initial estimate of the solution and improve this approximation over iterations.

⌑ **Gauss-Seidel method**

⌑ **Jacobi method**

⌑ **Conjugate gradient method**

V.4 Gaussian Method (Gaussian Elimination)

The Gaussian method (or Gaussian elimination) is a direct method which is based on three main steps:

- **Triangulation** : The matrix A is transformed into an upper triangular matrix U .
- **Resolution by backward substitution** : Once U is obtained, the unknowns are calculated starting with the last one.
- **Partial or full pivot (optional)** : To improve accuracy, swap lines to maximize pivot at each step.

Gaussian elimination method The Gaussian elimination method allows to find the solution of the system (AX=B) by transforming it into an upper triangular system which has the same solution. This method therefore consists of eliminating all the terms below the diagonal of the matrix (A).

✓ Augmented matrix

The augmented matrix of the linear system (AX = B) is the matrix of dimension (n) by (n + 1) that we obtain by adding the right-hand side (B) to the matrix (A), that is to say:

$$\left[\begin{array}{ccccc|c} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} & b_2 \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} & b_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} & b_n \end{array}\right]$$

Since elementary operations must be performed on both the rows of the matrix (A) and the rows of the vector (B), this notation is very useful.

The Gaussian elimination method is applied only in the case where no row permutation is performed.

In practice, the Gaussian elimination method consists of two parts:

1. Transformation of the system into another triangular system having the same solution, by applying, on the lines, the following operations:

$$L_i \leftarrow L_i - \left(\frac{a_{ik}}{a_{kk}}\right) L_k \ , \qquad k = 1:n-1 \ , \qquad i = k+1:n$$

The index (k) is the step number, varies from 1 to (n− 1). The index (i) is the line number (Li): For each value of (k) and (i) varies from (k + 1) to (n)

We then obtain an upper triangular system of the form:

$$\begin{bmatrix} a'_{11} & a'_{12} & a'_{13} & \cdots & a'_{1n} \\ 0 & a'_{22} & a'_{23} & \cdots & a'_{2n} \\ 0 & 0 & a'_{33} & \cdots & a'_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & a'_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b'_1 \\ b'_2 \\ b'_3 \\ \vdots \\ b'_n \end{bmatrix}$$

Where $a'_{ij} = 0$ for all $i > j$

2- Calculate the solution (x_i) of the triangular system (V-13)

$$x_i = \begin{cases} \dfrac{b'_n}{a'_{nn}} & si \quad i = n \\ \dfrac{\left(b'_i - \sum_{j=i+1}^{n} a'_{ij}\, x_j\right)}{a'_{ii}} & si \quad i \neq n \end{cases} \quad , \ (i = n, \ldots, 2, 1) \qquad \text{(V-14)}$$

Exercise 1:

Solve the following system of equations using the Gaussian elimination method:

$$\begin{bmatrix} 2 & 1 & 2 \\ 6 & 4 & 0 \\ 8 & 5 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 26 \\ 35 \end{bmatrix}$$

Solution to exercise 1:

We first write the augmented matrix:

$$\left[\begin{array}{ccc|c} 2 & 1 & 2 & 10 \\ 6 & 4 & 0 & 26 \\ 8 & 5 & 1 & 35 \end{array}\right]$$

We then perform operations on its lines in order to obtain an upper triangular system. We follow these steps:

Step 1: Elimination on column 1 of the matrix

We apply elementary operations on lines 2 and 3 as follows:

$$\left[\begin{array}{ccc|c} 2 & 1 & 2 & 10 \\ 6 & 4 & 0 & 26 \\ 8 & 5 & 1 & 35 \end{array}\right] \quad \begin{array}{l} (L_2 \leftarrow L_2 - (6/2)L_1) \\ (L_3 \leftarrow L_3 - (8/2)L_1) \end{array}$$

We obtain:

$$\begin{bmatrix} 2 & 1 & 2 & | 10 \\ 0 & 1 & -6 & |-4 \\ 0 & 1 & -7 & |-5 \end{bmatrix}$$

Note **:** element 2 is called *pivot* .

Step 2: Elimination on column 2 of the matrix

We apply an elementary operation on line 3 as follows:

$$\begin{bmatrix} 2 & 1 & 2 & | 10 \\ 0 & 1 & -6 & |-4 \\ 0 & 1 & -7 & |-5 \end{bmatrix} \quad (L_3 \leftarrow L_3 - (1/1)L_2)$$

We obtain:

$$\begin{bmatrix} 2 & 1 & 2 & | 10 \\ 0 & 1 & -6 & |-4 \\ 0 & 0 & -1 & |-1 \end{bmatrix}$$

Note **:** element 1 is called *pivot* .

The upper triangular system obtained is therefore:

$$\begin{bmatrix} 2 & 1 & 2 \\ 0 & 1 & -6 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10 \\ -4 \\ -1 \end{bmatrix}$$

Or again:

$$\begin{array}{rcl} 2x_1 + 1x_2 + 2x_3 & = & 10 \\ (0)x_1 + 1x_2 + (-6)x_3 & = & -4 \\ (0)x_1 + (0)x_2 + (-1)x_3 & = & -1 \end{array}$$

So :

$$x_3 = -1/-1 = 1\,, x_2 = \frac{-4 - (-6)(1)}{1} = 2\,, x_1 = \frac{10 - (1)(2) - (2)(1)}{2} = 3$$

V.5 Gauss-Seidel method

The Gauss-Seidel method is an iterative method based on successive improvement of the solution. It follows the following steps:

1. We rearrange the system so that A is **strictly diagonal dominant** or **positively defined** , guaranteeing convergence.
2. We initialize x $_0$ (initial approximate solution).

3. At each iteration k , each unknown is updated as follows:

$$X^{k+1} = TX^k + C$$

Iterative methods used to solve a linear system of the form (V-2) are often written in the form:

$$\max\left|X^{k+1} - X^k\right| \leq \varepsilon$$

or (x^{k+1}) is the solution vector calculated at iteration (k+ 1), the matrix (T) and the vector (C) depend on the method in question. Indeed, by this relation we calculate the vector (x^{k+1}) from the vector (x^k) starting with a given starting vector (x^0). We stop the iterations when the following convergence condition is satisfied:

Or ε is the precision of the desired solution. That is to say, we stop the iterations when all the elements of the vector (x^{k+1}) converge to a precision ε The last vector (x^{k+1}) calculated will therefore be the solution.

The Gauss-Seidel method is just a special case of iterative methods.

If we assume for the moment that all the elements of the diagonal are non-zero ($a_{ii} \neq 0 \ \forall \ i$) From an initial approximation of the solution that we will note $X_0 = [x_1^0, x_n^0]^T$ we construct the algorithm

$$x_1^{k+1} = \frac{1}{a_{11}}\left(b_1 - \sum_{j=2}^{n} a_{1j}x_j^k\right)$$

$$x_2^{k+1} = \frac{1}{a_{22}}\left(b_2 - \sum_{j=1, j\neq 2}^{n} a_{2j}x_j^k\right)$$

$$x_3^{k+1} = \frac{1}{a_{33}}\left(b_3 - \sum_{j=1, j\neq 3}^{n} a_{3j}x_j^k\right)$$

$$\vdots$$

$$x_n^{k+1} = \frac{1}{a_{nn}}\left(b_n - \sum_{i=1}^{n-1} a_{nj}x_j^k\right)$$

Which consists of isolating the coefficient of the diagonal of each line of the system. More generally, we write :

$$x_i^{k+1} = \frac{1}{a_{ii}}\left(b_i - \sum_{j=1, j\neq i}^{n} a_{ij}x_j^k\right)$$

Which can also be expressed:

$$x_i^{k+1} = \frac{1}{a_{ii}}\left(b_i - \sum_{j=1}^{i-1} a_{ij}x_j^k - \sum_{j=i+1}^{n} a_{ij}x_j^k\right)$$

- **Convergence of the Gauss-Seidel method Definition** A matrix is said to be *strictly diagonally dominant* if:

$$|a_{ii}| > \sum_{j=1, j\neq i}^{n} |a_{ij}| \quad , \quad \forall i$$

This definition means that the diagonal term (a_{ii}) of the matrix (A) is clearly dominant since its absolute value is greater than the sum of the absolute values of all the other terms of the row.

Convergence theorem: If the matrix A is strictly diagonally dominant, the Gauss-Seidel method converges, whatever the initial solution (X^0)

V.6.3 Exercice V.2

Soit le système linéaire suivant :

$$\begin{bmatrix} 3 & 1 & -1 \\ 1 & 5 & 2 \\ 2 & -1 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 17 \\ -18 \end{bmatrix} \qquad \text{(V-33)}$$

1) Vérifier que la méthode de Gauss-Seidel converge pour ce système.
2) Résoudre le système (V-33) par la méthode de Gauss-Seidel, à une précision de 10^{-3}, en utilisant le vecteur de départ : $X^0 = [0 \quad 0 \quad 0]^T$.

V.6.4 Corrigé d'exercice V.2

1) On utilise la définition (V-32) et le théorème de convergence, on a :

$$|3| > |1| + |-1| \;, \qquad |5| > |1| + |2| \;, \qquad |-6| > |2| + |-1|$$

Les trois inégalités sont vérifiées, alors la matrice A est à diagonale strictement dominante, et par conséquent la méthode de Gauss-Seidel converge.
2) Pour le système (V-33), la méthode de Gauss-Seidel s'écrit :

$$x_1^{k+1} = \frac{1}{3}\left(2 - x_2^k + x_3^k\right)$$
$$x_2^{k+1} = \frac{1}{5}\left(17 - x_1^{k+1} - 2x_3^k\right)$$
$$x_3^{k+1} = \frac{1}{-6}\left(-18 - 2x_1^{k+1} + x_2^{k+1}\right)$$

partant de $[0 \quad 0 \quad 0]^T$, on trouve d'abord :

$$x_1^1 = \frac{1}{3}(2 - 0 + 0) = \frac{2}{3}$$
$$x_2^1 = \frac{1}{5}\left(17 - \frac{2}{3} - 0\right) = \frac{49}{15}$$
$$x_3^1 = \frac{1}{-6}\left(-18 - 2\left(\frac{2}{3}\right) + \frac{49}{15}\right) = \frac{241}{90}$$

tandis qu'à la deuxième itération on trouve :

$$x_1^1 = \frac{1}{3}\left(2 - \frac{49}{15} + \frac{241}{90}\right) = 0{,}4703704$$

$$x_2^1 = \frac{1}{3}\left(17 - 0{,}4703704 + 2\left(\frac{241}{90}\right)\right) = 2{,}234815$$
$$x_3^1 = \frac{1}{-6}(-18 - 2(0{,}4703704) + 2{,}234815) = 2{,}784321$$

ainsi que les itérations suivantes :

k	x_1^k	x_2^k	x_3^k
1	0.6666667	3.266667	2.677778
2	0.4703704	2.234815	2.784321
3	0.8498354	2.116305	2.930561
4	0.9380855	2.040158	2.972669
5	0.9775034	2.015432	2.989929
6	0.9914991	2.005729	2.996212
7	0.9968277	2.002150	2.998584
8	0.9988115	2.000804	2.999470
9	0.9995553	2.000301	2.999802
10	0.9998335	2.000113	2.999926

Tableau V-1 : Solutions obtenues par la méthode de Gauss-Seidel (Corrigé d'exercice V.2)