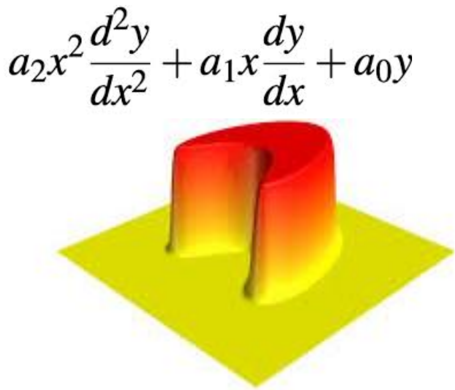
Chapter IV:

Numerical Solution of Ordinary Differential Equations



Introduction:

Ordinary differential equations (ODEs) play a fundamental role in the mathematical modeling of complex phenomena in physics, chemistry, biology, engineering, and many other fields. However, it is often impossible to solve these equations analytically, especially when the systems are nonlinear or involve complex conditions. In this context, numerical methods offer powerful tools for obtaining approximate solutions with satisfactory accuracy.

Among the most commonly used techniques, the Euler and Runge- Kutta methods occupy an important place. These approaches allow the differential equation to be discretized, thus transforming a continuous problem into a series of discrete calculations easily performed by computer.

- **Euler's method,** simple and intuitive, provides a first approach to understanding the basics of numerically solving ODEs. Although sometimes limited in accuracy and stability, it offers an overview of the fundamental principles.
- **Kutta method** offers more precise and stable solutions by introducing intermediate steps in the calculation. It is often used in practical applications requiring a compromise between efficiency and accuracy.

The aim of this chapter is to calculate the solution y(t) on the interval I = [a, b] of the Cauchy problem

$$y'(t)=f(t, y(t)) \quad \text{with} \quad y(t_0)=y_0$$

Thus, we understand that a differential equation is an equation depending on a variable t and a function (t) and which contains derivatives of (t). It can be written as:

$$y^{(k)}(t)=\frac{d^k y(t)}{dt^k}$$

Furthermore, it should be noted that very often the analytical solution does not exist, and we must therefore approximate the exact solution (t) by numerical methods.

IV.1. Euler's method:

__Leonhard Euler__ (in German), born on April 15, 1707 in Basel (Switzerland) and died on September 7, 1783 , was a mathematician and physicist. Swiss , who spent most of his life in the Russian Empire and Germany. He was notably a member of the Royal Prussian Academy of Sciences

Euler's method is one of the simplest numerical methods for solving ordinary differential equations (ODEs) of the form:

1. Principle of Euler's Method

The main idea is based on approximating the behavior of y(x) by a tangent, using the known derivative f(x , y) to calculate a linear progression. The solution is advanced incrementally over a given step size h .

In order to reach the solution y(t) , on the interval [a, b] , we choose n+1 dissimilar points , $t_0, t_1, \ldots\ldots\ldots\ldots t_n$ with $t_0 = a$ and $t_n = b$ and the discretization step is defined by:

h =(b - a)/2 .

The solution to be estimated can be approximated by a Taylor series expansion:

$$y(t_k+h)=y(t_k)+\frac{dy(t_k)}{dt}(t_{k+1}-t_k)+\ldots\ldots\ldots\ldots\ldots$$

With:

$$\frac{dy(t_k)}{dt}=f(t_k,y(t_k)) \quad \text{and} \quad h=t_{k+1}-t_k$$

This gives us Euler's numerical scheme

$$\begin{cases} y_{k+1}=y_k+f(t_k,y(t_k)) \quad \text{with} \quad k=0,1,2,\ldots\ldots\ldots\ldots n-1 \\ y=\text{initial value} \end{cases}$$

This method is first-order, meaning that the error is proportional to the square of the discretization step size (h). Intuitively, it's clear that to improve the accuracy of this method, it will suffice to reduce h. This reduction in the discretization step size will have the following incidence the increase in calculation time (- 1/h)

Furthermore, the advantage of Euler's method originates from the fact that it only requires the evaluation of the function for each integration step.

Kutta methods:

Carl David Tolmé Runge *(1856–1927) was a mathematician and physicist . German .*

Kutta was born in Pitschen , province of Silesia (now Byczyna in Poland).

Kutta type methods allow for greater accuracy (they generate numerical solutions closer to analytical solutions).

For the case of the **Runge- Kutta method of order 2** (also called the midpoint method, it calculates the average slope in two steps).

The approximate solution is given by:

$$\left\{ y_{i+1} = y_i + \frac{h}{2}(k_1 + k_2) \right.$$

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_{i+1}, y_i + hk_1)$$

Runge- Kutta method (Classical) of order 4, is a very popular explicit method. It calculates the value of the function at four intermediate points according to:

$$\begin{cases} y_0 \\ y_{k+1=} \quad y_k + \frac{h}{2} f\left(t_k + \frac{h}{2}, y_{2k}\right) + 2f\left(t_k + \frac{h}{2}, y_{3k}\right) + f(t_{k+1}, y_{4k}) \\ k = 1, . n - 1 \end{cases}$$

$$\begin{cases} y_{1k} = y_k \\ y_{2k} = y_k + \frac{h}{2}(f(t_k, y_{1k})) \\ y_{3k} = y_k + \frac{h}{2}(f(t_k + \frac{h}{2}, y_{2k})) \\ y_{4k} = y_k + h(f(t_k + \frac{h}{2}, y_{3k})) \end{cases}$$

Note that the number of terms retained in the Taylor series defines the order of the **Runge- Kutta method.** It follows that the **Runge- Kutta method** of order 4, stops at the end of the Taylor series.

Issue:

Consider the following differential equation:

$$y' = y - \frac{2x}{y}$$

$$y(0) = 1$$

Calculate the approximate solution of this equation at x = 0.6 using the Runge- Kutta method of order 2 by subdividing the interval into n = 3 parts with a precision of 4 digits after the decimal point.