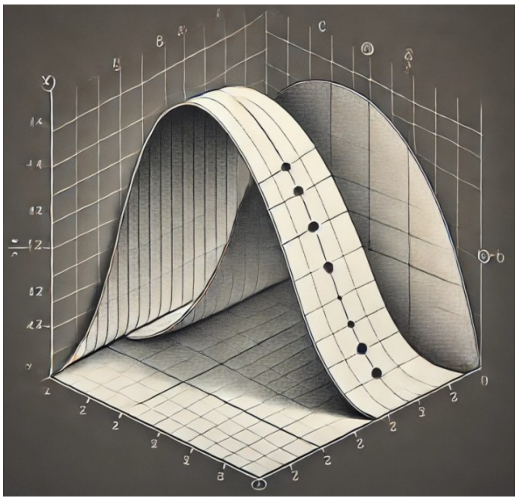
Chapter II:
Numerical Integration of Functions



Introduction:

Numerical integration involves approximating the value of a definite integral $\int f(x)dx$, when the function **f** is complex or when finding its antiderivative analytically is difficult. Below is a detailed explanation of the methods requested for numerical integration: the rectangle method, the trapezoidal method, and Simpson's method.

II.1 Rectangle Method

The rectangle method is a simple technique to estimate the integral of a continuous function f over an interval [a , b]. The idea is to divide this interval into n subintervals of equal width and then approximate the area under the curve using rectangles.

Formula:

Let [a , b] be the interval of integration and n be the number of subintervals (rectangles). The width of each subinterval is: $h = \frac{b-a}{n}$

The rectangle method can be applied in three variations, depending on the position of the point used to evaluate the function in each subinterval:

- **Left endpoint**: Uses the value of the function at the left end of each subinterval:

$$S = \sum_{i=1}^{n} (X_{i+1} - X_i) f(X_i)$$

- **Right endpoint**: Uses the value of the function at the right end of each subinterval:

$$S=\sum_{i=1}^{n}(X_{i+1}-X_i)f(X_{i+1})$$

- **Midpoint**: Uses the value of the function at the midpoint of each subinterval:

$$S=\sum_{i=1}^{n}(X_{i+1}-X_i)f\left(\frac{X_{i+1}+X_i}{2}\right)$$

II.2 Trapezoidal Method

The trapezoidal method is more precise than the rectangle method. It involves approximating the function f with a linear function over each subinterval $[x_i,x_{i+1}]$, then calculating the area of the resulting trapezoids.

Formula:

Let n be the number of subintervals of width h:

$$h=\frac{b-a}{n}$$

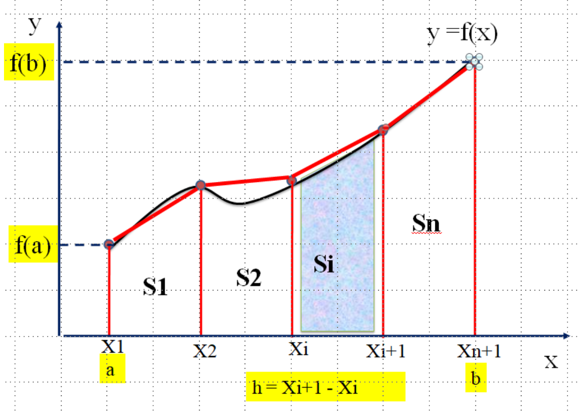
The points of subdivision are $x_1=a$, $x_2=a+h$, $x_n=b$

The approximate integral is given by:

$$S=\int_a^b f(x)dx \qquad S\approx\sum_{i=1}^{n}S_i$$

$$S\approx\sum_{i=1}^{n}(x_{i+1}-x_i)\left[f(x_i)+\left(\frac{f(x_{i+1})+f(x_i)}{2}\right)\right]$$

$$S\approx\frac{h}{2}\sum_{i=1}^{n}(f(x_{i+1})+f(x_i))$$



Here, we sum the function values at the intermediate points and multiply by two because each point (except the endpoints) is shared by two adjacent trapezoids.

II.3 Simpson's Method (Thomas- Simpson)

Thomas Simpson (20 August 1710 – 14 May 1761) was a British mathematician and inventor known for the eponymous Simpson's rule to approximate definite integrals.

Simpson's method is even more accurate than the trapezoidal method. It involves approximating the function **f** with a quadratic polynomial over each subinterval, providing a better approximation of the area under the curve.

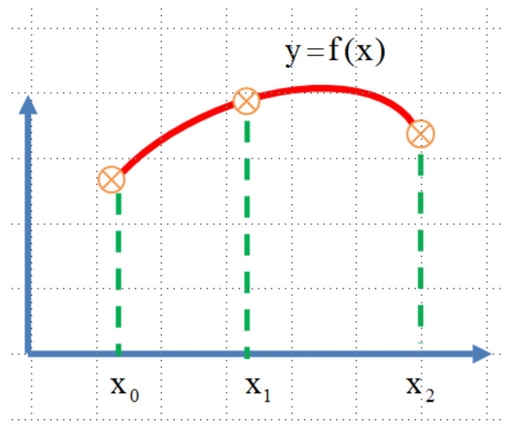
Formula:

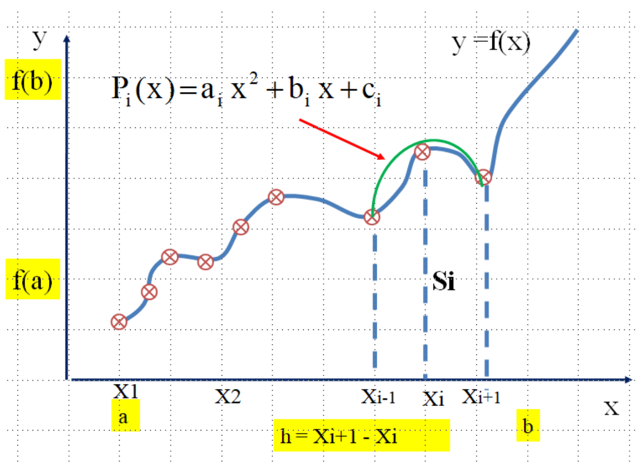
$$S=\frac{h}{3}\left[y_i+y_{i+1}+4\sum_{i(odd)}^{n} y_i+2\sum_{i(even)}^{n} y_i\right]$$

For this method, the number of subintervals n must be even:

- The coefficient 4 is applied to the terms where the index is odd.
- The coefficient 2 is applied to the terms where the index is even.
- The terms at the endpoints (f(a) and f(b)) are included once.

This method is more accurate because it uses a parabola to interpolate the function f over each subinterval, which better captures the curve of f.





$$\int_{x_0}^{x_1} f = \frac{h}{3}\left[f(x_0) + 4f(x_1) + f(x_2)\right]$$

$$h = x_1 - x_0 = x_2 - x_1$$

EXERCICE N°1:

Soit l'intégrale: $I = \int_1^4 (x^4 + 8x^3 - 5x^2 - 7x + 4)\,dx$

Calculer la valeur approchée de l'intégrale I par la méthode de Simpson en subdivisant l'intervalle en n = 8 sous-intervalles avec une précision de 4 chiffres après la virgule.

EXO1 :

$$I = \int_1^4 (x^4 + 8x^3 - 5x^2 - 7x + 4)\,dx.$$

$n = 8$, $h = \frac{b-a}{n} = \frac{4-1}{8} = 0.375$, $x_{i+1} = x_i + h$

La méthode de Simpson :

$$I = \frac{h}{3}\left[f(x_0) + f(x_n) + 4\sum_{\substack{i=1\\ \text{impair}}}^{n-1} f(x_i) + 2\sum_{\substack{i=2\\ \text{pair}}}^{n-2} f(x_i)\right].$$

$$I = \frac{h}{2}\left[f(x_0) + f(x_8) + 4\left(f(x_1) + f(x_3) + f(x_5) + f(x_7)\right) + 2\left(f(x_2) + f(x_4) + f(x_6)\right)\right].$$

i	0	1	2	3	4	5	6	7	8
x_i	1	1.375	1.75	2.125	2.5	2.875	3.25	3.625	4
$f(x_i)$	1	9.29321	28.6914	63.70336	119.3125	200.9768	314.6289	466.67602	664

$$I = \frac{0.375}{2}\left[1 + 664 + 4(9.29321 + 63.70336 + 200.9768 + 466.67602) + 2\left[28.6914 + 119.3125 + 314.6289\right]\right] = 569.10789.$$

exact = 596.1

$\pi - 0$ ... 0.314159 , $f(x) = \sin x^2$