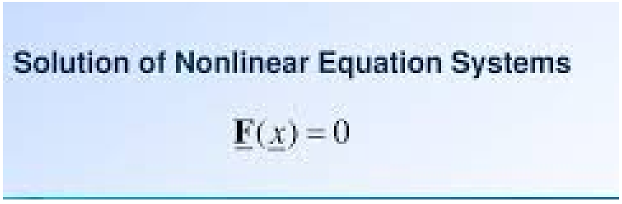
Chapter II:

Numerical Solution of Non-Linear Equations



Introduction:

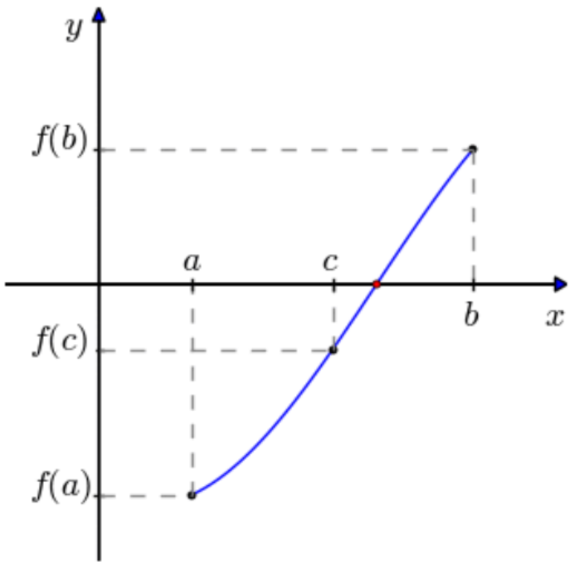
In numerical analysis, solving non-linear equations is a common problem that arises in various scientific and engineering fields. A non-linear equation is one where the unknown variable appears with exponents or is part of a transcendental function (such as trigonometric, logarithmic, or exponential functions). Two commonly used methods to solve non-linear equations are the **Bisection Method** and **Newton's Method**.

II.1 Bisection Method

The Bisection Method is a root-finding method that works by repeatedly dividing an interval in half and selecting the subinterval in which a root lies. This method guarantees convergence if the function is continuous on the interval and changes sign (i.e $f(a)\times f(b)\prec 0$)

Steps:

1. **Initial guess**: Start with an interval [a, b], where the function changes sign, meaning f(a) and f(b) have opposite signs: $f(a)\times f(b)\prec 0$
2. **Midpoint calculation**: Compute the midpoint: $c=\dfrac{b+a}{2}$
3. **Test the midpoint**: Evaluate f(c).
   - If f(c) = 0, then c is the root.
   - If $f(c)\neq 0$ to decide the next interval.
     - If $f(a)\times f(c)\prec 0$, the root lies in the left subinterval, so set b=c
     - If $f(c)\times f(b)\prec 0$, the root lies in the right subinterval, so set a=c



4. **Repeat**: Continue the process until the interval is sufficiently small, or the function value at the midpoint is close to zero.

- **Convergence criterion**

The convergence criterion refers to the conditions under which an iterative method can be considered to have reached a solution. For the dichotomy (bisection) method, the criterion is typically based on the size of the interval becoming smaller than a specified tolerance (ε), or the function value at the midpoint approaching zero within a given precision.

- **Stopping condition**

For the dichotomy method, the stopping condition depends on the desired precision (ε) and the initial interval.

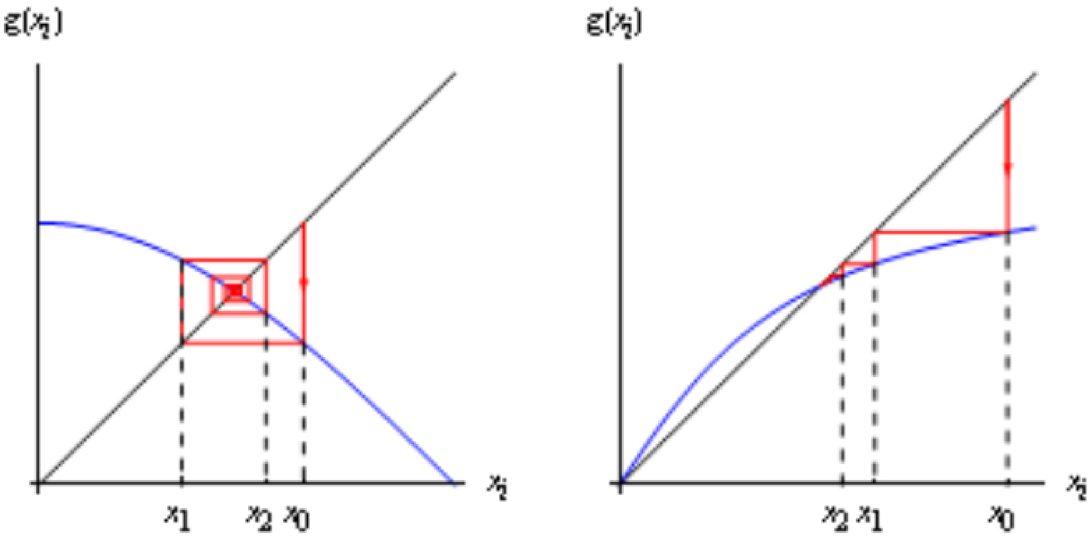
$$|b-a| \prec \varepsilon$$

- The numbers of iterations:

$$N \geq \frac{\log(b-a)-\log(\varepsilon)}{\log(2)}$$

II.2 Fixed-Point Method:

The **fixed-point method** is a numerical method for solving nonlinear equations of the form f(x) =0. To use this method, we rewrite the equation in the form x=g(x), where g is a function chosen such that the solutions of the equation correspond to fixed points of g. That is, for the solution x^*, we must have $g(x^*) = x^*$



Principle:

1. **Rewrite the equation**: Start by expressing the given equation f(x)=0 in the form x=g(x).
2. **Initial guess**: Select an initial approximation x_0.
3. **Iterative process**: Compute the sequence $x_{n+1} = g(x_n)$ until the difference $|x_{n+1} - x_n|$ becomes smaller than a specified tolerance epsilon ε.

4. **Convergence**: If the process converges, x_n approaches the solution of the equation.

Conditions for Convergence:

For the fixed-point method to converge, the function g(x) must be continuous and satisfy certain conditions in a neighborhood around the fixed point, such as:

- $|g'(x)| < 1$ near the solution.

Example

II.3 Newton-Raphson method

The Newton-Raphson method is a numerical technique used to find approximate solutions to nonlinear equations. It is an iterative approach that refines an initial guess to get closer to the exact solution.

■ **Principle of the Method**

The core idea of the Newton-Raphson method is to use the tangent line to the curve of the function to estimate the root of the equation. Given a function f(x) and the goal of solving f(x) = 0, the Newton-Raphson method uses the following approximation to refine an initial guess x_0:

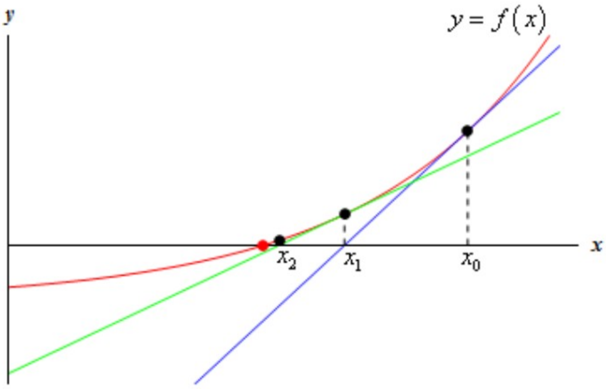


Figure I.3 : Principe de la méthode de Newton

Fig: **Principle of the Newton-Raphson method**

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Where:

- x_n is the current estimate of the root
- $f'(x_n)$ is the derivation of the function $f(x_n)$ evaluated at x_0

The iteration continues until the difference between x_n and x_{n+1} is small enough, indicating that a sufficiently accurate solution has been found.

The method converges quickly, and after a few iterations, you get a very accurate approximation of the actual solution.

Conclusion

The Newton-Raphson method is highly efficient for equations where the derivative is easy to compute and where you have an initial guess close to the root. It is widely used in various scientific and technical fields to solve nonlinear equations.