Exercise Series 01

Exercise 1:

Let F be a twice continuously differentiable function (C^2) from $\mathbb{R}^n$ to $\mathbb{R}$ satisfying

$$\lim_{\|x\|\to+\infty} F(x) = +\infty$$

and let $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$. Consider the Cauchy problem

$$\begin{cases} \dfrac{d}{dt}x(t) &= -\nabla F(x(t)), \quad t \in \mathbb{R}, \\ x(t_0) &= x_0, \quad x_0 \in \mathbb{R}^n. \end{cases} \tag{1}$$

1. Show that equation (1) admits a unique maximal solution defined on an interval $]T_-, T_+[$ of $\mathbb{R}$.

Exercise 2:

Consider the Riccati equation

$$\begin{cases} \dfrac{d}{dt}x(t) = x^2(t), \quad t \in \mathbb{R}, \\ x(0) = x_0, \quad x_0 \in \mathbb{R}. \end{cases} \tag{2}$$

1. Show that (2) admits a maximal solution defined on an interval $J \subset \mathbb{R}$ containing $t_0 = 0$.

2. Let $y(\cdot)$ be a continuous function defined on J satisfying

$$y(t) \le e^{\int_0^t y(s)ds}, \quad \forall t \in J.$$

Using question (1), show that:

$$y(t) \le \frac{1}{1-t}, \quad \forall t \in J.$$

3. Now consider the non-homogeneous Riccati equation

$$\begin{cases} \dfrac{d}{dt}x(t) = x^2(t) + t^2, \quad t \in \mathbb{R}, \\ x(0) = 0. \end{cases} \tag{3}$$

(3.1) Let $x(\cdot)$ be the maximal solution of (3). Define

$$z(t) = e^{-\int_0^t x(s)ds}.$$

Write a second-order differential equation for $z(\cdot)$ and show that

$$z(0) = z''(0) = z^3(0) = 0.$$

(3.2) Solve the differential equation in $z(\cdot)$ by seeking a solution in the form of a power series.

(3.3) Deduce that $x(\cdot)$ is defined on an interval $]-a, a[$ with $a \in]2, +\infty[$.

Exercise 3:

Consider the Cauchy problem

$$\begin{cases} \dfrac{d}{dt}x(t) = 2te^{-t^2} + \dfrac{x^5(t)}{1+x^4(t)}\cos(te^{x(t)}), \quad t \in \mathbb{R}, \\ x(0) = 1. \end{cases} \tag{4}$$

1. Show that (4) admits a unique maximal solution $x(\cdot)$ defined on an interval $]T_-, T_+[$ of $\mathbb{R}$ containing $t_0 = 0$.

2. Let $\phi(\cdot)$ be the maximal solution of (4). Using the integral form of (4), show that:

$$|\phi(t)| \leq 2 + \int_0^t |\phi(s)|ds, \quad \forall t \in J.$$

3. Deduce that:

$$\phi(t) \leq 2e^{|t|}, \quad \forall t \in J.$$

Exercise 4:

1. Find the maximal solutions of the following Cauchy problems:

$$x'(t) = x^3(t),\ x(0) = 0 \quad \text{and} \quad x'(t) = \frac{1}{x(t)},\ x(0) = 1.$$

2. Show that every maximal solution of the Cauchy problem

$$x'(t) = t\sqrt{t^2 + x^2(t)}, \quad x(t_0) = x_0$$

   is global.

Exercise 5:

Consider the differential equation:

$$x'(t) = f(t, x(t)),$$

where $f : \mathbb{R}^2 \to \mathbb{R}$ is a C^1 function. Suppose that ϕ and ψ are two solutions of this equation such that there exists $t_0 \in \mathbb{R}$ with

$$\phi(t_0) < \psi(t_0).$$

Show that:

$$\phi(t) < \psi(t), \quad \forall t \in \mathbb{R}.$$

Hint: Use a proof by contradiction.

Exercise 6:

Consider the differential equation

$$x'(t) = (a - x(t))(b - x(t)),$$

where a and b are real constants such that $a \leq b$.

1. Show that for every fixed initial condition $x_0 \in \mathbb{R}$, this equation admits a unique maximal solution $x(\cdot)$ such that $x(0) = x_0$.

2. What is this solution when $x_0 = a$ or $x_0 = b$?

3. Suppose that $a = b$.

   Find all solutions of the differential equation. Sketch the qualitative behavior of these solutions as a function of a and x_0.

4. Suppose $a = 1,\ b = 2,\ x_0 \neq a$, and $x_0 \neq b$.

   Determine the solution in terms of $x(0) = x_0$ and sketch its qualitative behavior as a function of x_0.

Exercise 7:

Consider the Cauchy problem

$$\begin{cases} \dfrac{d}{dt}x(t) = (1 + \cos t)x(t) - x^3(t), & t \in \mathbb{R}, \\ x(0) = x_0, \quad x_0 \in \mathbb{R}_+^*. \end{cases} \tag{5}$$

1. Show that (5) admits a unique local maximal solution ϕ defined on an interval $J \subset \mathbb{R}$ and that ϕ is of class C^2 on J.

2. Show that if there exists $t_1 \in J$ such that $\phi(t_1) = 0$, then $\phi(t) = 0$ for all $t \in J$.

3. Show that there exists $C > 0$ such that for $t \in J$,

$$0 < \phi(t) \leq \phi(0)e^{Ct}.$$

4. Show that the maximal solution ϕ is global.

5. Let $\psi(t, x)$ be the flow associated with problem (5) at $t = 0$. Define

$$P(x) = \psi(2\pi, x), \quad \text{for } x \in \mathbb{R}_+.$$

   Verify that $P(0) = 0$ and $P'(0) = e^{2\pi}$, then deduce that P satisfies the differential equation

$$P'(x) = e^{-4\pi}\left(\frac{P(x)}{x}\right)^3.$$

6. Solve this differential equation, then deduce that ϕ is 2π-periodic.