Chapter 3

Notion of Stability

In this chapter, we study the behavior of solutions of an ODE. Generally speaking, a solution is said to be *stable* if solutions starting from nearby initial values remain close to the considered solution for all future times. For linear differential systems, the stability of solutions is governed by the sign of the real parts of the eigenvalues of the matrix associated with the linear part of the ODE.

We conclude this chapter with a brief introduction to Lyapunov functions, which, under certain conditions, are very useful tools for studying the stability of ODEs.

3.1 Generalities: Stability of Linear Differential Systems

Consider the ODE:

$$y'(t) = f(t, y(t)), \tag{3.1}$$

where $f : \mathbb{R}_+ \times \Omega \to \mathbb{R}^n$ is continuous on $\mathbb{R}_+ \times \Omega$ and $\Omega \subset \mathbb{R}^n$ is an open set. We further assume that f is locally Lipschitz on Ω. Suppose that the ODE (3.1) has a global solution:

$$\varphi : \mathbb{R}^+ \to \Omega.$$

Definition 3.1.1.

The solution φ of (3.1) is said to be *stable* if the following two conditions are satisfied:

(i) For every $a \geq 0$, there exists $\mu(a) > 0$ such that for every $\xi \in B(\varphi(a), \mu(a))$, the maximal solution y of the Cauchy problem

$$\begin{cases} y' = f(t, y), \\ y(a) = \xi \end{cases} \tag{3.2}$$

is defined on $[a, +\infty)$.

(ii) For every $a \geq 0$ and every $\epsilon > 0$, there exists $\delta(a, \epsilon) \in [0, \mu(a))$ such that for every $\xi \in B(\varphi(a), \delta(a, \epsilon))$, the maximal solution of the Cauchy problem (3.2) satisfies:

$$\forall t \in [a, \infty), \quad \|y(t) - \varphi(t)\| \leq \epsilon.$$

Condition (i) of the above definition ensures that for any $a \geq 0$, there exists a unique solution of the Cauchy problem (3.2) defined on $[a, \infty)$, provided that ξ is sufficiently close to $\varphi(a)$. Moreover, condition (ii) guarantees that for any $\epsilon > 0$ and for ξ sufficiently close to $\varphi(a)$, the graph of the solution y of the Cauchy problem (3.2) (whose existence is ensured by condition (i)) remains sufficiently close to the graph of the solution φ.

Example 3.1.1. Consider the ODE $y' = 0$. We study the stability of the zero solution, i.e.,

$$\varphi : \mathbb{R}_+ \to \mathbb{R}, \quad \varphi \equiv 0.$$

• $a \geq 0$: for any $\xi \in \mathbb{R}$, the unique maximal solution of the Cauchy problem

$$\begin{cases} y' = 0, \\ y(a) = \xi \end{cases}$$

is defined on $[a, +\infty)$ and satisfies $y \equiv \xi$. Therefore, $\mu(a)$ can be chosen arbitrarily. Condition (i) of Definition 3.1.1 is satisfied.

• $\epsilon > 0$ fixed but arbitrary. We have:

$$\forall t \geq a, \ |y(t) - \varphi(t)| = |y(t)| \leq \epsilon \Longrightarrow |\xi| \leq \epsilon.$$

It follows that we can choose $\delta(a, \epsilon) = \epsilon$. Hence, condition (ii) of Definition 3.1.1 is satisfied.

Conclusion: the zero solution is stable.

Example 3.1.2. Consider the ODE $y' = y$. We study the stability of the zero solution.

• Let $a \geq 0$. The unique maximal solution of the Cauchy problem

$$y' = y, \quad y(a) = \xi$$

is defined on $[a, +\infty)$ and is given by

$$y(t) = \xi e^{t-a}.$$

Thus, $\mu(a)$ can be chosen arbitrarily, so condition (i) of Definition 3.1.1 is satisfied.

• However, condition (ii) is not satisfied since

$$\lim_{t \to +\infty} |y(t)| = +\infty.$$

Conclusion: the zero solution is unstable.

Other concepts of stability can also be defined.

Definition 3.1.2. The solution φ of (3.1) is said to be *uniformly stable* if it is stable and the constants $\mu(a)$ and $\delta(a, \epsilon)$ in Definition 3.1.1 do not depend on a and ϵ.

Definition 3.1.3. The solution φ of (3.1) is said to be *asymptotically stable* if it is stable and, for every $\xi \in B(\varphi(a), \mu(a))$, the maximal solution y of the Cauchy problem (3.2) satisfies

$$\lim_{t \to +\infty} \|y(t) - \varphi(t)\| = 0.$$

The notions of stability defined above are relative to a particular solution of the ODE (3.1). Therefore, two different solutions of the same ODE may differ in their stability properties.

Example 3.1.3. Consider the ODE, called the population dynamics model:

$$y' = by(p - y),$$

where $b > 0$. We study the stability of the solutions $\varphi_1 \equiv 0$ and $\varphi_2 \equiv p$.

• Let $a \geq 0$ and $\xi \in \mathbb{R}$. The Cauchy problem

$$\begin{cases} y' = by(p - y), \\ y(a) = \xi \end{cases}$$

admits a maximal solution defined on $[a, +\infty)$ given by

$$y(t) = \frac{p\xi e^{bp(t-a)}}{p + \xi(e^{bp(t-a)} - 1)}.$$

Condition (i) of Definition 3.1.1 is satisfied.

• We notice that

$$\lim_{t \to +\infty} |y(t)| = p^2 \neq 0.$$

Thus, condition (ii) of Definition 3.1.1 is not satisfied. Conclusion: the zero solution φ_1 is unstable. As for the solution φ_2, we have

$$\lim_{t \to +\infty} |y(t) - \varphi_2(t)| = \lim_{t \to +\infty} \left| \frac{p\xi e^{bp(t-a)}}{p + \xi(e^{bp(t-a)} - 1)} - p \right| = 0.$$

Conclusion: φ_2 is asymptotically stable.

Consider the ODE (3.1) and let $\varphi : \mathbb{R}^+ \to \Omega$ be a global solution of (3.1). The change of unknown function

$$y = x - \varphi$$

leads to the following ODE:

$$x'(t) = f(t, x(t) + \varphi(t)) - \varphi'(t). \tag{3.3}$$

In this chapter, we are only interested in the stability of the zero solution of the ODE (3.1). For this purpose, we assume that $0 \in \Omega$ and $\forall t \in \mathbb{R}_+, f(t, 0) = 0$.

Considering the zero solution, we rewrite Definitions 3.1.1, 3.1.2 and 3.1.3 as follows:

Definition 3.1.4. The zero function of the ODE (3.1) is said to be stable if:

(i) For all $a \geq 0$, there exists $\mu(a) > 0$ such that for every $\xi \in B(0, \mu(a))$, the maximal solution y of (3.2) is defined on $[a, +\infty)$.

(ii) For all $\epsilon > 0$, there exists $\delta(a, \epsilon) \in [a, \mu(a))$ such that for every $\xi \in B(0, \delta(a))$, the maximal solution of the Cauchy problem (3.2) satisfies:

$$\forall t \in [a, +\infty), \quad \|y(t)\| \leq \epsilon.$$

Definition 3.1.5. The zero solution of (3.1) is said to be *uniformly stable* if it is stable and the constants $\mu(a)$ and $\delta(a, \epsilon)$ in Definition 3.1.4 do not depend on a and ϵ.

Definition 3.1.6. The zero solution of (3.1) is said to be *asymptotically stable* if it is stable and for every $\xi \in B(0, \mu(a))$, the maximal solution y of the Cauchy problem (3.2) satisfies:

$$\lim_{t \to +\infty} \|y(t)\| = 0.$$

Next, we establish a connection between equilibrium points of an ODE and the stability of the zero solution.

Definition 3.1.7. A point $x^* \in \Omega$ is called a *stationary point* or *equilibrium point* of the ODE (3.1) if:

$$\forall t \in \mathbb{R}_+, \quad f(t, x^*) = 0.$$

It is clear that if x^* is a stationary point of the ODE (3.1), then the constant function $y \equiv x^*$ is a constant solution of the ODE (3.1), called a *stationary solution*. Moreover, it is clear that the identically zero solution is also a stationary solution of the ODE (3.1).

We assume that the ODE (3.1) is autonomous, i.e., the function f does not depend on t. We have the following result.

Theorem 3.1.1. *Let $f : \Omega \to \mathbb{R}^n$ be continuous, and let $y : [a, +\infty) \to \Omega$ be a solution of the ODE*

$$y'(t) = f(y(t)). \tag{3.4}$$

If

$$\lim_{t \to +\infty} y(t) = y^* \quad \text{and} \quad y^* \in \Omega,$$

then y^* is an equilibrium point of the ODE (3.4).[5]

Proof.

Let $f = (f_1, f_2, \dots, f_n)$. Suppose $y = (y_1, y_2, \dots, y_n)$ is a solution of the ODE (**??**). Consider a component y_i where $i \in \{1, 2, \dots, n\}$ defined on an interval $[m, m+1]$ with $m \in \mathbb{N} \cap [a, +\infty)$. Apply the mean value theorem to y_i on $[m, m+1]$.
Then, there exists a real number $\theta_m \in (m, m+1)$ such that:

$$y_i(m+1) - y_i(m) = y_i'(\theta_m) = f_i(y(\theta_m)).$$

Since

$$\lim_{m \to +\infty} \Big(y_i(m+1) - y_i(m) \Big) = 0,$$

and

$$\lim_{m \to +\infty} f_i(y(\theta_m)) = f_i(y^*),$$

we deduce that

$$\lim_{t \to +\infty} f_i(y(t)) = f_i(y^*)$$

for all $i \in \{1, 2, \dots, n\}$. Thus, $f(y^*) = 0$.

□

3.1.1 Stability of Linear Systems

We first study the simplest case, namely a linear system without forcing term:

$$y'(t) = \mathcal{A} y(t), \tag{3.5}$$

where $\mathcal{A} \in M_{n \times n}(\mathbb{C})$. The previous theorem will be useful.

Theorem 3.1.2. *The zero solution of the ODE (3.1) is stable (asymptotically stable, uniformly stable) if and only if every maximal solution of the ODE (3.5) is stable (asymptotically stable, uniformly stable).[5]*

Proof.

Suppose $y = \varphi$ is a maximal solution of the ODE (3.5). Using the change of variable $x = y - \varphi$, we deduce that φ corresponds to the maximal solution $x = 0$.
Thus, it suffices to notice that φ satisfies the conditions of Definitions 3.1.1, 3.1.2, and 3.1.3 if and only if the zero solution $x = 0$ satisfies the conditions of Definitions 3.1.4, 3.1.5, and 3.1.6.

□

By the previous theorem, all maximal solutions of the ODE (3.5) have the same type of stability. Hence, we will speak of the stability of the entire differential system.

To study the stability of the ODE (3.5), recall that for any $t_0 \geq 0$, the solution of the Cauchy problem

$$\begin{cases} y' = \mathcal{A}y, \\ y(t_0) = \xi \end{cases}$$

is defined on $[t_0, +\infty)$ and given by

$$y(t) = e^{(t-t_0)\mathcal{A}}\xi.$$

Thus, condition (i) in Definition 3.1.4 is satisfied. To understand what happens, consider the particular case: $n = 1$ and $\mathcal{A} = (a)$ where $a \in \mathbb{C}$.

$$\forall t \in [t_0, +\infty), \quad \|y(t)\| = e^{(t-t_0)\mathrm{Re}(a)}.$$

Hence, condition (ii) in Definition 3.1.5 is satisfied if and only if $\mathrm{Re}(a) \leq 0$.
Conclusion: the zero solution of the ODE $y' = ay$ is stable if and only if $\mathrm{Re}(a) \leq 0$. Moreover,

$$\lim_{t \to +\infty} |y(t)| = 0$$

if and only if $\mathrm{Re}(a) < 0$.
Conclusion: the zero solution of the ODE $y' = ay$ is asymptotically stable if and only if $\mathrm{Re}(a) < 0$. The following theorem characterizes the stability of solutions of the ODE (3.5).

Theorem 3.1.3. *Let λ_i, $i \in \{1, 2, \ldots, n\}$, be the eigenvalues of the matrix $\mathcal{A}$. Then the ODE (3.5) is:*

(i) Stable if and only if for every $i \in \{1, 2, \ldots, n\}$ either $\mathrm{Re}(\lambda_i) < 0$ or $\mathrm{Re}(\lambda_i) = 0$ and the Jordan block corresponding to λ_i is diagonalizable.

(ii) Asymptotically stable if and only if $\mathrm{Re}(\lambda_i) < 0$ for all $i \in \{1, 2, \ldots, n\}$.

[5]

Proof.

We distinguish the following cases:

• **The matrix $\mathcal{A}$ is diagonalizable:**

In this case, after a linear change of coordinates, $\mathcal{A}$ reduces to the matrix

$$\overline{\mathcal{A}} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix},$$

where $\lambda_1, \dots, \lambda_n$ are the eigenvalues of $\mathcal{A}$. Thus, the ODE (3.5) reduces to the linear ODEs:

$$y_j' = \lambda_j y_j, \quad y_j(t_0) = \xi_j, \quad j \in \{1, 2, \dots, n\}.$$

The solutions are

$$y_j = \xi_j e^{\lambda_j (t-t_0)}, \quad j \in \{1, 2, \dots, n\}.$$

The solutions are stable if and only if $\mathrm{Re}(\lambda_j) \leq 0$ for all j, and asymptotically stable if and only if $\mathrm{Re}(\lambda_j) < 0$ for all j.

• **The matrix $\mathcal{A}$ is non-diagonalizable:**

In this case, we treat each block in a triangularization of $\mathcal{A}$. Specifically, suppose

$$\begin{pmatrix} \lambda & \cdots & * \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda \end{pmatrix} = \lambda I + N,$$

where N is a nonzero upper-triangular nilpotent matrix. Then

$$e^{(t-t_0)\mathcal{A}} = e^{(t-t_0)\lambda I} e^{(t-t_0)N} = e^{\lambda(t-t_0)} \sum_{k=0}^{n-1} \frac{(t-t_0)^k}{k!} N^k.$$

The result follows from the properties of the Jordan blocks and the growth of $e^{(t-t_0)\mathcal{A}}$.

□

3.1.2 Stability in the Sense of Lyapunov

Consider the ODE

$$y' = f(t, y), \tag{3.6}$$

where $f : \mathbb{R}_+ \times \Omega \to \mathbb{R}^n$ with Ω an open subset of $\mathbb{R}^n$ containing 0. We assume that f satisfies the following two conditions:

(i) f is continuous on $\mathbb{R}_+ \times \Omega$ and locally Lipschitz on Ω.

(ii) $f(t, 0) = 0$, $\forall t \in \mathbb{R}_+$.

The first condition guarantees the existence and uniqueness of a solution to the Cauchy problem associated with the ODE (3.6). The second condition shows that the function $\varphi \equiv 0$ is a solution of the ODE (3.6).

Definition 3.1.8. A function $V : \mathbb{R}_+ \times \Omega \to \mathbb{R}_+$ is said to be positive definite on $\mathbb{R}_+ \times \Omega$ if there exists a continuous, increasing function $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\omega(r) = 0$ if and only if $r = 0$ and

$$\forall (t, x) \in \mathbb{R}_+ \times \Omega, \quad V(t, x) \geq \omega(\|x\|). \tag{3.7}$$

A function $V : \mathbb{R}_+ \times \Omega \to \mathbb{R}_-$ is negative definite on $\mathbb{R}_+ \times \Omega$ if $-V$ is positive definite on $\mathbb{R}_+ \times \Omega$.

Definition 3.1.9. A function $V : \mathbb{R}_+ \times \Omega \to \mathbb{R}_+$ is called a Lyapunov function of the system (3.6) if it satisfies the following conditions:

(i) V is of class C^1 on $\mathbb{R}_+ \times \Omega$ and $\forall t \in \mathbb{R}_+, \ V(t, 0) = 0$.

(ii) V is positive definite on $\mathbb{R}_+ \times \Omega$.

(iii) For all $(t, x) \in \mathbb{R}_+ \times \Omega$,

$$\frac{\partial V}{\partial t}(t,x) + \sum_{i=1}^{n} f_i(t,x) \frac{\partial V}{\partial x_i}(t,x) \leq 0. \tag{3.8}$$

Theorem 3.1.4. *If the ODE (3.6) possesses a Lyapunov function, then the zero solution is stable.[5]*

Proof.

Let $a \in \mathbb{R}_+$, $\xi \in \Omega$, and $y : [a, T_m) \to \Omega$ be the unique maximal solution of the ODE (3.6) satisfying $y(a) = \xi$. We first show that if ξ is sufficiently small, then $T_m = +\infty$.
Define the function $g : [a, T_m) \to \mathbb{R}_+$ by

$$g(t) = V(t, y(t)),$$

where V is the Lyapunov function corresponding to the ODE (3.6). The function g is of class C^1 on $[a, T_m)$ and by (3.8) we have

$$\begin{aligned} g'(t) &= \frac{\partial V}{\partial t}(t, y(t)) + \sum_{i=1}^{n} f_i(t, y(t)) \frac{\partial V}{\partial x_i}(t, y(t)) \\ &\leq 0. \end{aligned} \tag{3.9}$$

Hence, g is decreasing on $[a, T_m)$, so that

$$\forall t \in [a, T_m), \quad g(t) \leq g(a) \implies V(t, y(t)) \leq V(a, \xi).$$

From (3.7), it follows that

$$\omega(\|y(t)\|) \leq V(a, \xi), \quad \forall t \in [a, T_m).$$

Let $\rho > 0$ such that $B(0, \rho) \subset \Omega$. Since $V(a, \cdot)$ is continuous at 0 and $V(a, 0) = 0$, there exists $r = r(a) \in (0, \rho)$ such that

$$\forall \xi \in B(0, r), \quad V(a, \xi) < \omega(\rho).$$

Thus,

$$\forall t \in [a, T_m), \quad \omega(\|y(t)\|) \leq \omega(\rho) \implies \|y(t)\| \leq \rho.$$

Since $B(0, \rho) \subset \Omega$ and y is maximal, we conclude that

$$\forall \xi \in \Omega, \ \|\xi\| \leq r(a) \implies T_m = +\infty.$$

Similarly, for any $a \geq 0$ and $\epsilon > 0$, there exists $\delta(a, \epsilon) > 0$ such that

$$\forall \xi \in \Omega, \ \|\xi\| \leq \delta \implies \|y(t)\| \leq \epsilon.$$

Hence, the zero solution is stable. The proof is complete.

□

Theorem 3.1.5. *If the ODE (3.6) possesses a Lyapunov function V and there exist two continuous, strictly increasing functions $\lambda, \eta : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\lambda(r) = 0$ if and only if $r = 0$, $\eta(s) = 0$ if and only if $s = 0$, and*

$$V(t, x) \leq \lambda(\|x\|), \quad \forall (t, x) \in \mathbb{R}_+ \times \Omega, \tag{3.10}$$

$$\frac{\partial V}{\partial t}(t, x) + \sum_{i=1}^{n} f_i(t, x) \frac{\partial V}{\partial x_i}(t, x) \leq -\eta(\|x\|), \quad \forall (t, x) \in \mathbb{R}_+ \times \Omega, \tag{3.11}$$

then the zero solution is asymptotically stable.[5]

Proof.

From Theorem 3.1.4, it follows that the zero solution is stable. The inequalities (3.10) and (3.11) further imply that the zero solution converges to zero as $t \to +\infty$, hence it is asymptotically stable.

□

3.1.3 Solved Exercises

• **Exercise 01:**

Find the equilibrium points of the differential equation

$$\frac{dx}{dt} = \sin x$$

Solution:

Equilibria satisfy $\sin(x) = 0$.

Thus, there is an infinite number of equilibria:

$$x_k^* = k\pi, \quad k \in \mathbb{Z}.$$

• **Exercise 02:**