Chapter 2

Ordinary Differential Equations under Constraint

We consider the ODE:

$$y'(t) = f(t, y(t)) \tag{2.1}$$

where $f : I \times \Omega \to \mathbb{R}^n,\ I \subset \mathbb{R},\ \Omega \subset \mathbb{R}^n$.

For various reasons (physical, chemical, $\cdots$), the solutions of the ODE (2.1) must satisfy certain requirements called "viability constraints."

The theory of viability aims to establish a link between the dynamics (in our case the ODE (2.1)) and a given constraint, which in this context is considered as a nonempty subset $\mathcal{K}$ of $I \times \Omega$. More precisely, we have the following definitions:

Definition 2.0.1.

(i) A solution y of the Cauchy problem

$$\begin{cases} y'(t) = f(t, y(t)), \\ y(t_0) = x \end{cases} \tag{2.2}$$

with $(t_0, x) \in \mathcal{K}$ is said to be viable with respect to the constraint "$\mathcal{K}$" if there exists $T > t_0$ such that $[t_0, T] \subset I$ and

$$\forall t \in [t_0, T], \ (t, y(t)) \in \mathcal{K}.$$

(ii) The constraint $\mathcal{K}$ is said to be viable with respect to (2.1) if for all $(t_0, x) \in \mathcal{K}$, the Cauchy problem (2.2) admits a viable solution $y : [t_0, T] \to \Omega$ with $[t_0, T] \subset I$.

Remark 2.0.1. In this chapter, we only consider C^1 solutions. Nevertheless, the viability of a solution can also be defined if it is of Carathéodory type or weak...

We note here that in the case where $\mathcal{K} = I \times \Omega$ and Ω is an open subset of $\mathbb{R}^n$, Peano's theorem guarantees that if f is continuous on $\mathcal{K}$, then for all $(t_0, x) \in \mathcal{K}$, there exists $T > t_0$ such that $[t_0, T] \subset I$ and a solution $y : [t_0, T] \to \Omega$ to the Cauchy problem $y' = f(t, y), \ y(t_0) = x$.
Thus, if Ω is an open set and f is continuous on $\mathcal{K}$, then $\mathcal{K}$ is viable with respect to the ODE (2.1).
In the case where Ω is not open, the previous result is no longer valid.
The following example will illustrate this possibility.

Example 2.0.1. We consider the Cauchy problem

$$y'(t) = f(t, y(t)) = e^{y(t)} \tag{2.3}$$

where $f : [0,1] \times [0,1] \to \mathbb{R}$, and f is continuous on $[0,1] \times [0,1]$. The unique solution y of the problem (2.3) satisfying $y(0) = 1$ is given by

$$\forall t \in [0,1], \ \ y(t) = e^t.$$

We immediately observe that $\forall t \in (0,1), \ \ y(t) \notin \Omega$. That is, the solution graph leaves (immediately after its start) the set $\mathcal{K} = [0,1] \times [0,1]$. Hence, $\mathcal{K}$ is not viable with respect to (2.3).

The first viability theorem was announced by Nagumo in 1942. In fact, Nagumo discovered the missing condition required to ensure the viability of a set that is not necessarily open. In order to understand it properly, we will focus in the next section on the notion of the tangent cone.

2.1 Bouligand-Severi Tangent Cone

The notion of a tangent vector to a set at a given point was introduced simultaneously and independently by Bouligand and Severi in 1932. In what follows, let Ω be a nonempty subset of $\mathbb{R}^n$. Recall that the distance between a point $v \in \mathbb{R}^n$ and the set Ω is defined by:

$$\operatorname{dist}(\eta; \Omega) = \inf_{\omega \in \Omega} \|\eta - \omega\|.$$

Definition 2.1.1. We say that the vector η is tangent to Ω at x if:

$$\lim_{h \to 0^+} \inf \frac{1}{h} \operatorname{dist}(x + hv, \Omega) = 0.$$

The set of all tangent vectors to Ω at x is denoted by $\mathcal{T}_\Omega(x)$.

Example 2.1.1. Let $\Omega = [a,b] \subset \mathbb{R}$. We compute the set $\mathcal{T}_\Omega(a)$. Two cases are distinguished:

- If $v \in \mathbb{R}_+$, then

$$\lim_{h\to 0^+} \inf \frac{1}{h} \text{dist}(a + hv, [a,b]) = 0.$$

It follows that

$$\mathbb{R}_+ \subset \mathcal{T}_\Omega(a).$$

- If $v \in \mathbb{R}_-^*$, then

$$\lim_{h\to 0^+} \inf \frac{1}{h} \text{dist}(a + hv, [a,b]) = \lim_{h\to 0^+} \frac{|a + hv - a|}{h} = |v| \neq 0.$$

Hence,

$$\mathcal{T}_\Omega(a) = \mathbb{R}_+.$$

Example 2.1.2. Let $\Omega = \{(x,y) \in \mathbb{R}^2,\ x^2 + y^2 = 1\} \subset \mathbb{R}^2$. We want to determine all tangent vectors η to Ω at $x = (1,0)$. Let $\eta = (\eta_1, \eta_2)$. Then:

$$\begin{aligned} \text{dist}(x + h\eta, \Omega) &= \sqrt{(1 + h\eta_1)^2 + (h\eta_2)^2} - 1 \\ &= \frac{2h\eta_1 + h^2\eta_1^2 + h^2\eta_2^2}{\sqrt{(1 + h\eta_1)^2 + (h\eta_2)^2} + 1}. \end{aligned} \tag{2.4}$$

Consequently,

$$\lim_{h\to 0} \inf \frac{1}{h} \text{dist}(x + h\eta, \Omega) = \eta_1.$$

Thus, the vector η is tangent to Ω at $x = (1,0)$ if and only if $\eta_1 = 0$. Hence we can write:

$$\mathcal{T}_\Omega((1,0)) = \{(\eta_1, \eta_2) \in \mathbb{R}^2,\ \eta_1 = 0\}.$$

Proposition 2.1.1. The set $\mathcal{T}_\Omega(x)$ is a closed cone.[16]

Proof.

Recall that $\mathcal{T}_\Omega(x)$ is a cone if:

- $0 \in \mathcal{T}_\Omega(x)$. Indeed,

$$\lim_{h\to 0^+}\inf \frac{1}{h}\text{dist}(x+hv,\Omega) = \lim_{h\to 0^+}\frac{1}{h}\text{dist}(x,\Omega) = 0.$$

- $\forall s \in \mathbb{R},\ sv \in \mathcal{T}_\Omega(x)$. If $v \in \mathcal{T}_\Omega(x)$ and $\forall s \in \mathbb{R}$, then:

$$\lim_{h\to 0^+}\inf \frac{1}{h}\text{dist}(x+hsv,\Omega) = \lim_{\theta\to 0^+}\inf s\frac{1}{\theta}\text{dist}(x+\theta v,\Omega) = 0.$$

  This proves that $sv \in \mathcal{T}_\Omega(x)$.

  It remains to show that $\mathcal{T}_\Omega(x)$ is closed. Let $(v_n)_n$ be a sequence in $\mathcal{T}_\Omega(x)$ converging to $\overline{v}$. For all $n \in \mathbb{N}$ and $h > 0$, we have:

$$\frac{1}{h}\text{dist}(x+h\overline{v},\Omega) \le \|v_n - \overline{v}\| + \frac{1}{h}\text{dist}(x+hv_n,\Omega). \tag{2.5}$$

  Since $v_n \to \overline{v}$, passing to the limit as $n \to +\infty$ gives:

$$\lim_{h\to 0^+}\inf \frac{1}{h}\text{dist}(x+h\overline{v},\Omega) = 0.$$

Thus the proof is complete.

□

Proposition 2.1.2. If x belongs to the interior of Ω, then $\mathcal{T}_\Omega(x) = X$.

Proof.

If x belongs to the interior of Ω, then there exists an open ball centered at x with radius r such that $B(x,r) \subset \Omega$. For any $v \in X$, we can choose h sufficiently close to 0 such that
$x + hv \in B(x,r)$. Indeed,

$$\|x + hv - x\| = h\|v\|.$$

To ensure $x + hv \in B(x,r)$, i.e., $h\|v\| < r$, it suffices to choose $h \in \left(0, \frac{r}{\|v\|}\right)$. Thus,

$$\lim_{h \to 0^+} \inf \frac{1}{h} \mathrm{dist}(x + hv, \Omega) = 0.$$

□

Proposition 2.1.3. Let $\Omega \subset \mathbb{R}^n$ and $x \in \Omega$. The following assertions are equivalent:

(i) v is tangent to Ω at x, $(v \in \mathcal{T}_\Omega(x))$.

(ii) For all $\delta, \epsilon > 0$, there exist $h \in (0, \delta)$ and $p_h \in B(0, \epsilon)$ such that

$$x + hv + hp_h \in \Omega.$$

(iii) There exist three sequences:

$$(h_n)_n \subset \mathbb{R}_+, \quad \lim_n h_n = 0,$$

$$(v_n)_n \subset \mathbb{R}^n, \quad \lim_n v_n = 0,$$

$$(p_n)_n \subset \mathbb{R}^n, \quad \lim_n p_n = 0,$$

such that $\forall n \in \mathbb{N},\ x + h_n v_n + h_n p_n \in \Omega$.

[16]

Proof.

(i)$\Longrightarrow$(ii): Suppose v is tangent to Ω at x. From Definition 2.1.1, we have:

$$\lim_{h\to 0^+} \inf \frac{1}{h}\text{dist}(x+hv,\Omega)=0.$$

This is equivalent to:

$$\forall\delta>0,\ \forall\epsilon>0,\ \exists h\in(0,\delta),\ \exists\omega_h\in\Omega,\ \frac{1}{h}\|x+hv-\omega_h\|\leq\epsilon.$$

Setting $p_h=\dfrac{-x-hv+\omega_h}{h}$, we obtain:

$$x+hv+hp_h=\omega_h\in\Omega,$$

with $p_h\in B(0,\epsilon)$.

(ii)$\Longrightarrow$(iii): it suffices to choose $\delta=\epsilon=\frac{1}{n},\ n=1,2,\ldots.$

(iii)$\Longrightarrow$(i): follows immediately from the definition.

□

2.2 Other Types of Tangent Cones

2.2.1 Bouligand-Severi Tangent Cone

Definition 2.2.1. Let $\Omega\subset\mathbb{R}^n$ and $x\in\Omega$. The **Bouligand-Severi tangent cone** (or contingent cone) at x is defined by:

$$T_\Omega(x)=\left\{v\in\mathbb{R}^n : \lim_{h\to 0^+}\frac{1}{h}\text{dist}(x+hv,\Omega)=0\right\}.$$

Example 2.2.1. Let $\Omega=\mathbb{R}^2_+=\{(x,y):x\geq 0,\ y\geq 0\}$ and $x=(0,0)$. Then:

$$T_\Omega(0,0)=\mathbb{R}^2_+.$$

2.2.2 Interior Tangent Cone

Definition 2.2.2. The **interior tangent cone** at $x \in \Omega$ is defined by:

$$T^i_\Omega(x) = \left\{ v \in \mathbb{R}^n : \exists\, h_k \downarrow 0^+,\, \exists\, v_k \to v \text{ with } x + h_k v_k \in \Omega \ \forall k \right\}.$$

Example 2.2.2. For $\Omega = \mathbb{R}^2_+$ and $x = (1,1)$ (interior point), we have:

$$T^i_\Omega(1,1) = \mathbb{R}^2.$$

For $x = (0,1)$ (boundary point):

$$T^i_\Omega(0,1) = \{(v_1, v_2) \in \mathbb{R}^2 : v_1 \geq 0\}.$$

2.2.3 Exterior Tangent Cone

Definition 2.2.3. The **exterior tangent cone** at $x \in \Omega$ is defined by:

$$T^e_\Omega(x) = \{v \in \mathbb{R}^n : -v \in T^i_\Omega(x)\}.$$

Example 2.2.3. For $\Omega = \mathbb{R}^2_+$ and $x = (0,0)$:

$$T^e_\Omega(0,0) = \mathbb{R}^2_- = \{(v_1, v_2) : v_1 \leq 0, v_2 \leq 0\}.$$

2.2.4 Clarke Tangent Cone

Definition 2.2.4. The **Clarke tangent cone** at $x \in \Omega$ is defined as the convex closure of the Bouligand cone:

$$T_\Omega^C(x) = \overline{\text{co}}\, T_\Omega(x).$$

Example 2.2.4. Let $\Omega = \{(x, y) \in \mathbb{R}^2 : y \geq |x|\}$ and $x = (0, 0)$. Then:

$$T_\Omega^C(0, 0) = \{(v_1, v_2) : v_2 \geq |v_1|\}.$$

2.2.5 Proximal Tangent Cone

Definition 2.2.5. The **proximal tangent cone** at $x \in \Omega$ is defined by:

$$T_\Omega^P(x) = \left\{ v \in \mathbb{R}^n : \exists\, \alpha > 0 \text{ such that } \langle v, y - x \rangle \leq \alpha \|y - x\|^2,\ \forall y \in \Omega \right\}.$$

Example 2.2.5. Let $\Omega = \{(x, y) : x^2 + y^2 \leq 1\}$ (unit disk) and $x = (1, 0)$. Then:

$$T_\Omega^P(1, 0) = \{(v_1, v_2) : v_1 \leq 0\}.$$

2.3 Comparison Between Types of Tangent Cones

Main Differences

- **Contingent cone (Bouligand-Severi)**: directions accessible by small perturbations. May be non-convex.

- **Interior cone**: directions where one immediately stays inside Ω. Always included in the contingent cone.
- **Exterior cone**: outward directions, defined as the opposite of the interior cone.
- **Clarke cone**: convex closure of the contingent cone, always convex.
- **Proximal cone**: defined by a quadratic condition, used in optimization.

Relations Between Cones

$$T_\Omega^P(x) \subseteq T_\Omega^i(x) \subseteq T_\Omega(x), \quad T_\Omega^C(x) = \overline{\text{co}}(T_\Omega(x)), \quad T_\Omega^e(x) = -T_\Omega^i(x).$$

Summary Table

Type of cone	**Definition**	**Relation**
Contingent $T_\Omega(x)$	$\liminf_{h\to 0^+} \frac{1}{h}\text{dist}(x+hv,\Omega) = 0$	General, not always convex
Interior $T_\Omega^i(x)$	Directions remaining in Ω right from the start	$T_\Omega^i(x) \subseteq T_\Omega(x)$
Exterior $T_\Omega^e(x)$	$T_\Omega^e(x) = -T_\Omega^i(x)$	Cone of outgoing directions
Clarke $T_\Omega^C(x)$	$\overline{\text{co}}(T_\Omega(x))$	Convex, larger than $T_\Omega(x)$
Proximal $T_\Omega^P(x)$	$\langle v, y-x\rangle \leq \alpha\|y-x\|^2$	$T_\Omega^P(x) \subseteq T_\Omega(x)$

2.4 Nagumo's Theorem

As previously mentioned, Nagumo discovered in 1942 the missing condition to ensure the viability of a set with respect to an ordinary differential equation (ODE). More precisely, we have the following result:

Theorem 2.4.1 (Nagumo). *Let $\Omega \subset \mathbb{R}^n$ be a closed set and*

$$f : I \times \Omega \longrightarrow \mathbb{R}^n$$

a continuous mapping.

A necessary and sufficient condition for

$$K = I \times \Omega$$

to be viable with respect to the ODE (1.22) is the ***tangential condition****:*

$$\forall (t_0, x) \in K, \quad f(t_0, x) \in T_\Omega(x),$$

where $T_\Omega(x)$ denotes the tangent cone (or contingent) to Ω at x.[17]

- Nagumo's theorem ensures that if $f : I \times \Omega \to \mathbb{R}^n$ is continuous and Ω is closed, then $K = I \times \Omega$ is viable with respect to (1.21) if and only if the tangential condition (TC) is satisfied.
- Nagumo's theorem generalizes the one established by **Peano** in 1890. Indeed, if Ω is open, the tangential condition becomes redundant.

2.4.1 Proof of Nagumo's Theorem

This paragraph is devoted to the proof of Nagumo's theorem.

Proof of the necessary condition

Let $f : I \times \Omega \to \mathbb{R}^n$. We need to show that if $\mathcal{K} = I \times \Omega$ is viable with respect to (2.2) then the tangential condition (TC) is satisfied.

Proof.

Let $(t_0, x) \in I \times \Omega$. Since $\mathcal{K} = I \times \Omega$ is viable with respect to (2.2), there exists $T > t_0$ such that $[t_0, T] \subset I$ and a solution $y : [t_0, T] \to \Omega$ of the Cauchy problem:

$$\begin{cases} y' = f(t, y) \\ y(t_0) = x. \end{cases}$$

A first-order Taylor expansion of y near t_0 gives:

$$y(t_0 + h) = y(t_0) + y'(t_0)(h) + o(h) = x + f(t_0, x)h + o(h), \;\; h \simeq 0^+.$$

Consequently,

$$\begin{aligned} \operatorname{dist}(x + f(t_0, x)h, \Omega) &= \operatorname{dist}(y(t_0 + h) - o(h), \Omega) \\ &\leq \operatorname{dist}(y(t_0 + h) - o(h), y(t_0 + h)) + \operatorname{dist}(y(t_0 + h), \Omega) \end{aligned} \tag{2.6}$$

Knowing that h is sufficiently close to 0, we have $y(t_0 + h) \in \Omega$ and thus:

$$\frac{1}{h}\operatorname{dist}(x + f(t_0, x)h, \Omega) \leq \frac{1}{h}(|o(h)| + 0) \leq |o(1)|.$$

Taking the limit as $h \to 0^+$:

$$\liminf_{h \to 0^+} \frac{1}{h}\operatorname{dist}(x + f(t_0, x)h, \Omega) = 0.$$

Conclusion: $f(t_0, x) \in \mathcal{T}_\Omega(x)$. Therefore, the necessary condition is satisfied.

□

Proof of the sufficient condition

We present a sketch of the proof of Nagumo's theorem.

Let $(t_0, x) \in I \times \Omega$. Consider the Cauchy problem:

$$\begin{cases} y' = f(t, y(t)) \\ y(t_0) = x. \end{cases} \tag{2.7}$$

We aim to construct, from the tangential condition (TC), a viable solution of the problem (2.7). In this part, the continuity of f is crucial, and the proof proceeds in three steps:

(i) The first step is to construct, from the tangential condition (TC), a sequence of approximate solutions to the Cauchy problem (2.7).

(ii) Show the convergence of the constructed sequence to a continuous function y.

(iii) Show that the limit function y is viable.

• **Step 1:** Construction of a sequence of approximate solutions to the Cauchy problem (2.7).

Let $(t_0, x) \in \mathcal{K}$ and consider the Cauchy problem (2.7). We fix in advance the domain of definition of the approximate solutions. In fact, the domain must be uniform for all solutions (it does not depend on the chosen approximation). This will help in the limit passage step.

Since Ω is closed and $x \in \Omega$, there exists $\rho > 0$ such that $B(x, \rho) \cap \Omega$ is closed.

The function f being continuous is thus bounded on $B(x, \rho) \cap \Omega$, so let:

$$M = \sup_{y \in B(x,\rho) \cap \Omega} \|f(t_0, y)\|. \tag{2.8}$$

We choose T such that:

$$(T - t_0)(M + 1) \leq \rho. \tag{2.9}$$

The following lemma is constructive and allows us to build a sequence of approximate solutions for (2.7).

Lemma 2.4.2.

Let T, ρ and $M > 0$ be fixed as above. For any $\epsilon > 0$, there exist three functions $\sigma : [t_0, T] \to [t_0, T]$ (Lebesgue integrable) and $y : [t_0, T] \to \mathbb{R}^n$ satisfying:

(i) $\sigma(t) \le t$ *and* $t - \sigma(t) \le \epsilon$, *for all* $t \in [t_0, T]$.

(ii) $\|g(t)\| \le \epsilon$, *for all* $t \in [t_0, T]$.

(iii) $y(\sigma(t)) \in B(x, \rho) \cap \Omega$ *for all* $t \in [t_0, T]$ *and* $y(T) \in B(x, \rho) \cap \Omega$.

(iv)

$$y(t) = x + \int_{t_0}^{t} f(\sigma(s), y(\sigma(s)))ds + \int_{t_0}^{t} g(s)ds, \ \text{for all } t \in [t_0, T].$$

[18]

!

Proof.

Initialization of the problem: This step initializes the construction of the functions in the previous lemma.

This means constructing the functions σ, g, and y on the interval $b$hile satisfying the lemma's statements.

Let $(t_0, x) \in \mathcal{K}$ and $\epsilon \in (0, 1)$. Since $f(t_0, x)$ is tangent to $\mathcal{K}$, there exist $\delta > 0, h \in (0, \delta)$, and $p \in X$ such that $\|p\| \le \epsilon$ and

$$x + f(t_0, x)h + hp \in \Omega,$$

we define the functions

$$\sigma : [t_0, t_0 + h] \to [t_0, t_0 + h], \ g : [t_0, t_0 + h] \to \mathbb{R}^n, \ \text{and } y : [t_0, t_0 + h] \to \mathbb{R}^n$$

as follows:

$$\begin{cases} \forall t \in [t_0, t_0 + h], \ \sigma(t) = t_0, \\ \forall t \in [t_0, t_0 + h], \ g(t) = p, \\ \forall t \in [t_0, t_0 + h], \ y(t) = x + (t - t_0) f(t_0, x) + (t - t_0) p. \end{cases}$$

We check that the functions σ, g, and y satisfy statements (i)–(iv) of the above lemma. Indeed, conditions (i), (ii), and (iv) are satisfied. We show the truth of condition (iii).

Using relations (2.8) and (2.9) we immediately deduce:

• **Step 1:** Now we show the existence of the functions σ, g, and y on the interval $[t_0, T]$. We define the set $\mathcal{S}$ of all triplets (σ, g, y) satisfying the lemma (2.4.2) on the interval $[t_0, t_0 + h] \subset [t_0, T]$. We equip $\mathcal{S}$ with a binary relation $\preceq$ defined as follows:

Let $(\sigma_1, g_1, y_1), (\sigma_2, g_2, y_2) \in \mathcal{S}$ defined respectively on $[t_0, t_0 + h_1]$ and $[t_0, t_0 + h_2]$. Then:

$$(\sigma_1, g_1, y_1) \preceq (\sigma_2, g_2, y_2) \Leftrightarrow \begin{cases} h_1 \leq h_2 \\ \forall t \in [t_0, t_0 + h_1], \ (\sigma_1(t), g_1(t), y_1(t)) = (\sigma_2(t), g_2(t), y_2(t)). \end{cases}$$

Thus, we show that the set $\mathcal{S}$ admits a maximal element $\overline{y} : [t_0, \overline{T}] \to \mathbb{R}^n$. The next step is to prove that $\overline{T} = T$. We reason by contradiction and assume that $\overline{T} < T$. The tangential condition is satisfied for all $(\tau, \xi) \in \mathcal{K}$, which allows us to construct a solution defined on $[\overline{T}, \overline{T} + \delta] \subset I$. The patching principle then allows us to define a solution of problem (2.7) on $[t_0, \overline{T} + \delta]$, which contradicts the maximality of $\overline{y}$.

• **Step 2:** By setting $\epsilon = \frac{1}{n}, \ n = 1, 2, \ldots,$ we construct a sequence of approximate functions (y_n). The Ascoli-Arzelà theorem plays a crucial role in showing that the sequence (y_n) converges uniformly to a function y solving problem (2.7).

• **Step 3:** We show that the solution $y : [t_0, T] \to \mathbb{R}^n$ is viable, i.e.

$$\forall t \in [t_0, T], y(t) \in \Omega.$$

□

2.4.2 Application

In this paragraph, we present some applications of viability theory.

Lipschitz dependence of solutions on initial conditions

Consider the ODE:

$$y' = f(y(t)) \tag{2.10}$$

where $f : \Omega \to \mathbb{R}^n$ and $\Omega \subset \mathbb{R}^n$ is open. We show that the solutions of ODE (2.10) depend in a Lipschitz manner on the initial conditions.

Theorem 2.4.3.

Assume that f is Lipschitz on Ω, i.e., there exists a constant $k > 0$ such that:

$$\forall (y,z) \in \Omega,\ \ \|f(y) - f(z)\| \leq k\|y - z\|.$$

Let $y_0, z_0 \in \Omega$. If $y : [t_0, T] \to \mathbb{R}^n$ is a solution of ODE (2.10) with $y(t_0) = y_0$, then there exists $z : [t_0, T] \to \mathbb{R}^n$ solution of ODE (2.10) with $z(t_0) = z_0$ such that:

$$\forall t \in [t_0, T],\ \ \|y(t) - z(t)\| \leq e^{k(T-t_0)}\|y_0 - z_0\|.$$

[5]

Proof.

We present a sketch of the proof. Consider the Banach space $X = \mathbb{R}^n \times \mathbb{R}$.
Let

$$\mathcal{K} = \{(y,z) \in X \mid \|y\| \leq z\}.$$

Consider the ODE:

$$u' = F(u), \tag{2.11}$$

where $F : X \to X$ is defined by:

$$\forall u \in X,\ F(u) = (f(y), z),\ u = (y, z).$$

The idea is to show that the set $\mathcal{K}$ is viable with respect to ODE (2.11).

□

Banach fixed-point theorem

We provide an extension of Banach's fixed-point theorem.

Theorem 2.4.4.
Let Ω be a closed subset of $\mathbb{R}^n$ and $g : \Omega \to \mathbb{R}^n$ a Lipschitz function with constant $L < 1$. If:

$$\forall x \in K,\ \lim_{h \to 0^+} \inf \frac{1}{h} \mathrm{dist}(x + h(f(x) - x), K) = 0,$$

then g has a unique fixed point. [?]

Proof.

Consider $F : K \to \mathbb{R}^n$ defined by:

$$\forall x \in K,\ F(x) = g(x) - x.$$

Consider the ODE:

$$u' = F(u). \tag{2.12}$$

We show that K is viable with respect to ODE (2.12). Let $T > 0$ and consider the operator $\mathbb{Q} : K \to K$ defined by:

$$\forall x \in K,\ \mathbb{Q}(x) = u(T).$$

We show that the operator $\mathbb{Q}$ is contractive, hence it admits a fixed point. A simple calculation gives $g(x) - x = 0$. The proof is thus complete.

□