Chapter 1

Generalization on Differential Equations

1.1 Notation of Solutions and Types of Solutions (Local, Maximal, Global, and Saturated Solutions)

Definition 1.1.1.

• **A differential equation** is a relation involving the derivatives of the unknown function.

• **A differential equation** in which the unknown function depends on a **single** variable is called an ordinary differential equation, abbreviated as **ODE**.[5]

Example 1.1.1.

The equations:

$$y'(t) = 3y^3(t) - 2y(t) + 1, \quad y''(t) = \cos(y) + 3$$

are ordinary differential equations, whereas the following are not.

Definition 1.1.2.

The **order of an ODE** is the order of the highest derivative appearing in the equation. More precisely, an ODE is said to be of order n if it can be written as:

$$F(t, y(t), y'(t), \dots, y^{(n)}(t)) = 0,$$

where F is a function defined on $D(F)$, a subset of $\mathbb{R} \times (\mathbb{R}^n)^{n+1}$, with values in $\mathbb{R}^n$.

Example 1.1.2.

The ODE:

$$3y' + y = t^2 + t \quad \text{is of order 1.}$$

$$\sin(y'') = t \quad \text{is of order 2.}$$

Remark 1.1.1.

Any ODE can be rewritten as a first-order ODE. For instance, if we have

$$F(t, y(t), y'(t), \dots, y^{(n)}(t)) = 0,$$

we can define an equivalent system in the form:

$$F(t, z_1(t), z_2(t), \dots, z_{n-1}(t), z'_{n-1}(t)) = 0.$$

Example 1.1.3.

Consider the polynomial ODE

$$y''(t) = y^3(t) - ay^2(t) + 2ty'(t), \quad t \in \mathbb{R}.$$

If we set

$$z(t) = (y(t), y'(t)) = (z_1(t), z_2(t)),$$

then

$$\begin{aligned} z_1'(t) &= y'(t) = z_2(t), \\ z_2'(t) &= y''(t) \\ &= y^3(t) - ay^2(t) + 2ty'(t) \\ &= z_1^3(t) - az_1^2(t) + 2tz_2(t). \end{aligned}$$

Thus, if we define

$$F(t, z_1(t), z_2(t)) = \Big(z_2(t),\ z_1^3(t) - az_1^2(t) + 2tz_2(t)\Big),$$

the ODE can be rewritten in the form

$$z'(t) = F(t, z(t)).$$

Definition 1.1.3.

An ODE is said to be a **linear ODE of order** n if it can be written as

$$a_n(t)y^{(n)}(t) + a_{n-1}(t)y^{(n-1)}(t) + \cdots + a_1(t)y'(t) + a_0(t)y(t) = b(t),$$

where a_i for $i = 0, 1, \ldots, n$ and b are functions defined on a subinterval of $\mathbb{R}$.

Example 1.1.4.

The ODE

$$y' + y = 2t$$

is linear, whereas

$$\sin(y'(t)) = 2$$

is nonlinear.

Definition 1.1.4.

An ODE is said to be **autonomous** if the independent variable does not explicitly appear in the equation. More precisely, an ODE of order n is autonomous if it can be written in the form

$$F\Big(y(t), y'(t), \ldots, y^{(n)}(t)\Big) = 0,$$

where F is a function defined on $D(F)$, a subset of $(\mathbb{R}^n)^{n+1}$, with values in $\mathbb{R}^n$.

Example 1.1.5.

The ODE: $y' + y = 2$ is autonomous, whereas the ODE: $y' + y = 2t$ is not autonomous.

1.1.1 Notions of Solution and Types of Solutions

We consider the ODE:

$$y' = f(t, y) \tag{1.1}$$

where $f : I \times \Omega \to \mathbb{R}^n,\ \Omega \subset \mathbb{R}^n$.

Definition 1.1.5.

(1) The function $y : J \to \Omega$ is a solution of equation (1.1), where J is a non-empty subinterval of I, if $J \subset I$.
- $\forall t \in J,\ y(t) \in \Omega$.
- y is differentiable on J and $\forall t \in J,\ y' = f(t, y)$.

(2) If in addition $y \in C^1(J)$, we say that y is a solution of class C^1.

Example 1.1.6. Determine the C^1 solutions of the ODE:

1. $y' = 3t^2 y$ admits infinitely many C^1 solutions defined on $\mathbb{R}$ by

$$y(t) = ce^{t^3}, \quad y(t) = c,$$

   where c is an arbitrary constant.

2. The ODE $y' = 1 + y^2$ admits infinitely many C^1 solutions defined on any interval of the form

$$]c - \frac{\pi}{2}, c + \frac{\pi}{2}[, \quad y(t) = \tan(t - c),$$

   where c is a constant.

It may happen that the domain of definition of a solution of equation (1.1) is different from I (since f is defined on $I \times \Omega$).

Definition 1.1.6. Let $y : J \mapsto \Omega$ be a solution of problem (1.1).

1. The solution y is called **global** if $J = I$. Otherwise ($J \subset I$), y is called **local**.
2. The solution y is called **maximal** if it admits no extension to another solution of problem (1.1).

- Reminder:

$$\tilde{y} : \tilde{J} \to \Omega \text{ is an extension of } \; y : J \mapsto \Omega \text{ if } J \subset \tilde{J} \text{ and } \forall t \in J, \; \tilde{y}(t) = y(t).$$

Example 1.1.7. Consider the Cauchy problem:

$$\begin{cases} y'(t) = -2t\, y^2(t), \quad t \in \mathbb{R}, \\ y(0) = 1. \end{cases}$$

1. Assume that $y(t) \neq 0,\ \forall t \in \mathbb{R}$. Then one can write

   $$\frac{y'(t)}{y^2(t)} = -2t.$$

   Integrating with respect to t, we obtain

   $$-\frac{1}{y(t)} - \left(\frac{-1}{y(0)}\right) = -t^2.$$

   Hence a solution is

   $$y(t) = \frac{y(0)}{t^2 y(0) + 1}.$$

   Since $y(0) = 1$, we get

   $$y(t) = \frac{1}{t^2 + 1}.$$

   Notice that $y(t)$ is well defined on $\mathbb{R}$, hence $J = \mathbb{R}$, i.e. the solution is **global**.

2. Consider the Cauchy problem:

   $$\begin{cases} y'(t) = 2t\, y^2(t), \\ y(0) = 1. \end{cases}$$

   It admits a solution of the form

   $$y(t) = \frac{1}{1 - t^2}, \quad t \in J =]-1, 1[.$$

   This solution is **maximal** but not global ($J \neq \mathbb{R}$).

Remark 1.1.2. Any solution $y : J \longmapsto \Omega$ of the ODE (1.1) such that

$$\lim_{t \to a} y(t) = \infty \quad \text{where } a = \sup J \text{ or } a = \inf J$$

cannot be extended to another solution of the same problem.

1.2 Qualitative Study of ODEs in Finite Dimension (Peano and Cauchy–Lipschitz Theorems)

Cauchy Problem and Integral Equation

Definition 1.2.1.

Consider the ODE

$$y'(t) = f(t, y(t)), \tag{1.2}$$

where $f : I \times \Omega \to \mathbb{R}^n$, I is a subinterval of $\mathbb{R}$, Ω is a subset of $\mathbb{R}^n$, and let $(t_0, y_0) \in I \times \Omega$.

- A **Cauchy problem** is the task of finding a solution of (1.2) satisfying the initial condition $y(t_0) = y_0$. It is written in the form:

$$\begin{cases} y'(t) = f(t, y(t)), \\ y(t_0) = y_0. \end{cases} \tag{1.3}$$

Definition 1.2.2.

A function $y : J \to \Omega$ is called a solution of the Cauchy problem (1.3), where J is a subinterval of I containing t_0, if $y(t_0) = y_0$ and y is a solution of the ODE (1.2).

Remark 1.2.1.

Let $(t_0, y_0) \in I \times \Omega$ and let y be a solution of the Cauchy problem (1.3).

The solution y is called a **right-hand solution** if

$$J = [t_0, T] \quad \text{or} \quad J = [t_0, T[.$$

Similarly, y is called a **left-hand solution** if

$$J = [T, t_0] \quad \text{or} \quad J =]T, t_0].$$

Otherwise, the solution y is called **bilateral** ($\inf J < t_0 < \sup J$).

Proposition 1.2.1.

Suppose that f is continuous on $I \times \Omega$.

Then a function $y : J \to \Omega$ is a solution of the Cauchy problem (1.3) if and only if y is **continuous** on J and

$$\forall t \in J, \quad y(t) = y_0 + \int_{t_0}^{t} f(s, y(s))\, ds.$$

[12]

Preuve 1.

Assume that y is a solution of the Cauchy problem (1.3). Since y is differentiable on J, it is therefore **continuous**. As f is continuous on $I \times \Omega$, the mapping $s \mapsto f(s, y(s))$ is also **continuous**.

The solution y satisfies:

$$\forall t \in J, \quad y'(t) = f(t, y(t)).$$

Integrating both sides from t_0 to t, we obtain:

$$\forall t \in J, \quad \int_{t_0}^{t} y'(s)\, ds = \int_{t_0}^{t} f(s, y(s))\, ds.$$

Since $y(t_0) = y_0$, it follows that:

$$\forall t \in J, \quad y(t) = y(t_0) + \int_{t_0}^{t} f(s, y(s))\, ds.$$

Conversely: Since f is continuous on $I \times \Omega$ and y is continuous on J, the mapping $s \mapsto f(s, y(s))$ is continuous on J.

Moreover, from the integral equation we have:

$$\forall t \in J, \quad y(t) = y_0 + \int_{t_0}^{t} f(s, y(s))\, ds.$$

Differentiating with respect to t, we get:

$$\forall t \in J, \quad y'(t) = f(t, y(t)).$$

This completes the proof.

Local Existence of Solutions for a Cauchy Problem

Theorem 1.2.2 (Peano's Theorem). *If f is continuous on $I \times \Omega$, where $\Omega \subset \mathbb{R}^2$ is open, then for every $(t_0, y_0) \in I \times \Omega$, the problem (1.3) admits **at least** one **local** solution $y : J \subset I \to \Omega$ defined on a subinterval $J \subset I$ containing t_0.[11]*

Proof. We present a proof method based on delay integral equations. ☐

Proposition 1.2.2. Consider the delay integral equation for $\lambda \geq 0$:

$$y_\lambda(t) = \begin{cases} x, & t \in [t_0 - \lambda, t_0] \\ x + \int_{t_0}^{t} f(s, y_\lambda(s-\lambda))ds, & t \in [t_0, t_0 + \delta]. \end{cases} \tag{1.4}$$

where $f : I \times \Omega \to \mathbb{R}^n$, $\Omega \subset \mathbb{R}^n$ is open and $(t_0, x) \in I \times \Omega$. Choose $\delta > 0$ such that $[t_0, t_0 + \delta] \subset I$ and $\lambda \geq 0$. Note that if $\lambda = 0$, then equation (1.4) reduces to the integral equation

$$y(t) = x + \int_{t_0}^{t} f(s, y(s))ds, \quad \forall t \in [t_0, t_0 + \delta],$$

which is exactly the integral equation associated with the Cauchy problem (1.3).

- **Proof Method:** The idea of the proof is as follows:

First, we show that the integral equation (1.4)

admits for every $\lambda > 0$, at least one solution $y_\lambda : [t_0 - \lambda, t_0 + \delta] \to \mathbb{R}^n$.

Next, by letting $\lambda \to 0$, we show that $(y_\lambda)_\lambda$ converges uniformly to a function y. Finally, we show that the limit y is a solution of the considered Cauchy problem.

We first consider the case $\Omega = \mathbb{R}^n$ and f is **continuous** on $I \times \mathbb{R}^n$. The following result will be useful.

Lemma 1.2.3. *If $f : I \times \mathbb{R}^n \to \mathbb{R}^n$ is* ***continuous****, then for every $(t_0, x) \in I \times \mathbb{R}^n$, $\lambda > 0$, and $\delta > 0$ such that $[t_0, t_0 + \delta] \subset I$, the equation (1.4) admits a* ***unique solution*** *defined on $[t_0 - \lambda, t_0 + \delta]$.*

[12]

Proof.

The construction of y_λ is done in two steps.

- Suppose $\delta \leq \lambda$. On the interval $[t_0 - \lambda, t_0]$, we have $y_\lambda \equiv x$. Now, for $t \in [t_0, t_0 + \delta]$, clearly if $s \in [t_0, t]$, then $s - \lambda \in [t_0 - \lambda, t_0]$, giving:

$$y_\lambda(s - \lambda) = x, \quad \forall s \in [t_0, t].$$

Consequently,

$$y_\lambda(t) = x + \int_{t_0}^{t} f(s, x)ds, \quad \forall t \in [t_0, t_0 + \lambda].$$

The construction is thus complete.

- Suppose $\lambda < \delta$. The construction of y_λ is done on $[t_0 - \lambda, t_0 + \lambda]$.
  - If $t_0 + \delta \leq t_0 + 2\lambda$, then for $t \in [t_0 + \lambda, t_0 + \delta]$,

$$s - \lambda \in [t_0, t_0 + \lambda], \quad \forall s \in [t_0 + \lambda, t],$$

so

$$y_\lambda(s - \lambda) = x + \int_{t_0}^{t} f(\theta, x)d\theta.$$

Hence, y_λ is defined on $[t_0 - \lambda, t_0 + \delta]$.

  - If $2\lambda \leq \delta$, repeat the same procedure.

After finitely many steps, there exists a natural number n such that $t_0 + n\lambda \geq t_0 + \delta$. This completes the construction of y_λ. Since y_λ is defined by integrals, it is continuous.

The proof is complete.

□

We have thus constructed a solution of the integral equation defined on $[t_0 - \lambda, t_0 + \delta]$ for an arbitrary choice of delay λ and any $\delta > 0$ such that $[t_0, t_0 + \delta] \subset I$.

The lemma below allows us to remove the delay $\lambda > 0$.

Lemma 1.2.4. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be continuous and bounded on $I \times \mathbb{R}^n$. Then for every $(t_0, x) \in I \times \mathbb{R}^n$ and $\delta > 0$ such that $[t_0, t_0 + \delta] \subset I$, the delay equation (1.4) has a* ***unique solution*** *defined on $[t_0, t_0 + \delta]$.[12]*

Proof.

Let $(t_0, x) \in I \times \mathbb{R}^n$ and $\delta > 0$ such that $[t_0, t_0 + \delta] \subset I$. Let $n \in \mathbb{N}$. Consider the delay integral equation with $\delta_n = \frac{\delta}{n+1}$:

$$y_n(t) = \begin{cases} x, & t \in [t_0 - \delta_n, t_0] \\ x + \int_{t_0}^{t} f(s, y_n(s - \delta_n))ds, & t \in (t_0, t_0 + \delta_n] \end{cases} \tag{1.5}$$

By Lemma 1.2.4, this integral equation admits a **unique solution**:

$$y_n : [t_0 - \delta_n, t_0 + \delta] \to \mathbb{R}^n, \quad \forall n \in \mathbb{N}.$$

Consider the family $\mathcal{F}$ of restrictions of the solutions y_n to the interval $[t_0, t_0 + \delta]$, also denoted y_n. To prove that the solutions y_n converge uniformly on $[t_0, t_0 + \delta]$, it suffices to apply the Arzelà-Ascoli theorem, which requires showing that the family $\mathcal{F}$ is equicontinuous and uniformly bounded.

The family $\mathcal{F}$ is uniformly bounded: We show that all functions y_n are bounded by a constant independent of n. Indeed, f is bounded on $I \times \mathbb{R}^n$ (assumption of the lemma). Thus,

$$\exists M > 0, \quad \forall (t, y) \in I \times \mathbb{R}^n, \quad \|f(t, y)\| \leq M.$$

From equation (1.5) we have:

$$\forall n \in \mathbb{N}, \ \forall t \in [t_0, t_0 + \delta], \quad \|y_n(t)\| \leq \|x\| + (t - t_0)M \leq \|x\| + \delta M.$$

Hence, the family $\mathcal{F}$ is uniformly bounded on $[t_0, t_0 + \delta]$.

The family $\mathcal{F}$ is equicontinuous: We observe that

$$\forall t, s \in [t_0, t_0 + \delta], \quad \|y_n(t) - y_n(s)\| \leq \left| \int_{s}^{t} \|f(\tau, y_n(\tau - \delta_n))\| d\tau \right| \leq M|t - s|.$$

This shows that $\mathcal{F}$ is equicontinuous on $[t_0, t_0 + \delta]$.

By the Arzelà-Ascoli theorem, $(y_n)_n$ converges uniformly to a continuous function

$y : [t_0, t_0 + \delta] \to \mathbb{R}^n$. Moreover,

$$\lim_{n\to\infty} (s - \delta_n) = s,$$

uniformly on $[t_0, t_0 + \delta]$. Since f is continuous on $I \times \mathbb{R}^n$, passing to the limit in (1.5) as $n \to +\infty$, we deduce:

$$\forall t \in [t_0, t_0 + \delta], \quad y(t) = x + \int_{t_0}^{t} f(s, y(s))ds,$$

so $y : [t_0, t_0 + \delta] \to \mathbb{R}^n$ is a solution of the Cauchy problem.

□

- We now move to the proof of Peano's main result.

Proof.

Let $(t_0, x) \in \Omega$. Since I and Ω are open, there exist $d > 0$ and $r > 0$ such that

$$[t_0 - d, t_0 + d] \subset I, \quad B(x, r) = \{\eta \in X; \|\eta - x\| \leq r\} \subset \Omega.$$

Define the function $\rho : \mathbb{R}^n \to \mathbb{R}^n$ by

$$\rho(y) = \begin{cases} y, & y \in B(x, r) \\ \dfrac{r}{\|y - x\|}(y - x) + x, & y \in \mathbb{R}^n \setminus B(x, r). \end{cases}$$

The map ρ is continuous and $\rho(\mathbb{R}^n) \subset B(x, r)$. Define now $g : (t_0 - d, t_0 + d) \times \mathbb{R}^n \to \mathbb{R}^n$ by

$$g(t, y) = f(t, \rho(y)), \quad \forall (t, y) \in (t_0 - d, t_0 + d) \times \mathbb{R}^n.$$

Since f is continuous, by Weierstrass' theorem its restriction to $(t_0 - d, t_0 + d) \times \mathbb{R}^n$ is bounded. By Lemma 1.2.4, for any $d' \in (0, d)$, the Cauchy problem

$$\begin{cases} y' = g(t, y) \\ y(t_0) = x \end{cases}$$

admits at least one solution $y : [t_0, t_0 + d'] \to \mathbb{R}^n$. Since y is continuous at t_0 and $y(t_0) = x$, for any $r > 0$, there exists $\delta \in (0, d')$ such that

$$\forall t \in [t_0, t_0 + \delta], \quad \|y(t) - x\| \leq r.$$

Hence, $g(t, y(t)) = f(t, y(t))$, so y is indeed a solution of the Cauchy problem. The proof is complete.

□

Remark 1.2.5.

(i) Continuity of f is not sufficient to guarantee uniqueness of the solution. The example below (constructed by G. Peano himself) illustrates this phenomenon.

(ii) The conditions of Peano's theorem are sufficient to guarantee the existence of at least one local solution to the Cauchy problem (1.3). It may happen that f is not continuous on $I \times \Omega$, yet the Cauchy problem (1.3) still admits at least one local solution (see exercises).

Example 1.2.1. It is easy to verify that the Cauchy problem

$$\begin{cases} y'(t) = 3y^{2/3} \\ y(0) = 0 \end{cases}$$

admits two solutions: the zero function $y_1 \equiv 0$ and the function $y_2 : \mathbb{R} \to \mathbb{R}$ defined by $y_2(t) = t^3$.

Cauchy–Lipschitz Theorem (Uniqueness of a Local Solution to a Cauchy Problem)

We are interested in the uniqueness of a local solution of problem (1.3) when it exists. As already noted in Example 1.2.1, the continuity of f is not sufficient to guarantee uniqueness.

For this purpose, additional conditions on the function f must be imposed.[5]

Definition 1.2.3.

The Cauchy problem (1.3) is said to have local uniqueness if for every $(t_0, x) \in I \times \Omega$, whenever u and v are two solutions of problem (1.3), there exists $\delta > 0$ such that $[t_0, t_0 + \delta] \subset I$ and

$$\forall t \in [t_0, t_0 + \delta],\ u(t) = v(t).$$

The Cauchy problem (1.3) has the property of global uniqueness if for every $(t_0, x) \in I \times \Omega$, all solutions of problem (1.3) coincide on the common part of their domains of definition.

Proposition 1.2.3.

The properties of local uniqueness and global uniqueness of a Cauchy problem are equivalent. [5, 2]

Preuve 2.

We need to prove that the property of local uniqueness implies global uniqueness (the reverse implication is trivial). Let $(t_0, x) \in I \times \Omega$. Consider two solutions $u : \mathbb{J} \to \Omega$ and $v : \mathbb{K} \to \Omega$ of problem (1.3). Define the set $\mathcal{C}(u, v)$ by

$$\mathcal{C}(u, v) = \{t \in \mathbb{J} \cap \mathbb{K} :\ u(s) = v(s),\ \forall s \in [t_0, t].\}$$

Since (1.3) has the property of local uniqueness, the set $\mathcal{C}(u, v)$ is nonempty. We show that $\mathcal{C}(u, v)$ is closed: let $(t_n)_n \subset \mathcal{C}(u, v)$ converge to i. Then $i \in \mathcal{C}(u, v)$. We must show that

$$\sup \mathcal{C}(u, v) = \sup(\mathbb{J} \cap \mathbb{K}).$$

For contradiction, assume $\sup \mathcal{C}(u, v) < \sup(\mathbb{J} \cap \mathbb{K})$. Let $b = \sup \mathcal{C}(u, v)$. Consider the Cauchy problem

$$\begin{cases} z' = f(t, z(t)), \\ z(b) = y(b). \end{cases}$$

The solution is defined on $[b, \delta] \subset I$. By the patching principle, we can thus define a solution of the Cauchy problem (1.3) on $[t_0, b + \delta]$, which contradicts the maximality of the solution. The proof is therefore complete.

Remark 1.2.6.
Since there is equivalence between the two uniqueness properties, we will henceforth simply say that the Cauchy problem has the uniqueness property whenever it applies.

Definition 1.2.4.
A function $f : I \times \Omega \to \mathbb{R}^n$ is said to be locally Lipschitz on Ω if for every compact set $\mathcal{K} \subset I \times \Omega$, there exists a constant $L = L(\mathcal{K}) > 0$ such that

$$\forall (t, x_1), (t, x_2) \in \mathcal{K}, \ \|f(t, x_1) - f(t, x_2)\| \leq L\|x_1 - x_2\|.$$

If the above condition holds on $I \times \Omega$, we say that the function f is Lipschitz on Ω.

Example 1.2.2.

1. Clearly, every Lipschitz function is locally Lipschitz.
2. The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(t, y) = |ty|$ is locally Lipschitz on $\mathbb{R}$.

   Indeed,

   $$|f(t, y_1) - f(t, y_2)| = |t||y_1 - y_2| \leq L(t)|y_1 - y_2|, \ \forall t \in [a, b] \times \mathbb{R}^n.$$
3. The function $f : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ defined by $f(t, y) = \sqrt{ty}$ is not locally Lipschitz on $\mathbb{R}_+$.

Remark 1.2.7.

If $f : I \times \Omega \to \mathbb{R}$ satisfies the Cauchy condition on Ω, i.e., for all $i, j = 1, 2, \ldots, n$ the partial derivatives $\dfrac{df_j}{dx_i}$ are continuous on $I \times \Omega$, then f is locally Lipschitz on Ω.

Proposition 1.2.4.

By virtue of Proposition 1.2.3, it suffices to prove that the Cauchy problem (1.3) has the property of local uniqueness. Let $(t_0, x) \in I \times \Omega$ and let $u : \mathbb{J} \to \Omega,\ v : \mathbb{K} \to \Omega$ be two solutions of problem (1.3). Since $I \times \Omega$ is open, there exist $\rho > 0$ and $d > 0$ such that

$$a + d \leq \sup(\mathbb{J} \cap \mathbb{K}), \text{ and } B(x, \rho) \subset \Omega.$$

The functions u and v are continuous at t_0. We may therefore choose δ such that

$$\forall t \in [t_0, t_0 + d[, \|u(t) - v(t)\| \leq \int_{t_0}^{t} \|f(s, u(s)) - f(s, v(s))\| ds \leq \int_{t_0}^{t} \|u(s) - v(s)\| ds.$$

Gronwall's Lemma allows us to conclude that

$$\forall t \in [t_0, t_0 + d[, \ \|y(t) - z(t)\| = 0.$$

Consequently,

$$\forall t \in J', \ y(t) = z(t).$$

[5, 2].

Theorem 1.2.8.

If the function $f : I \times \Omega \to \mathbb{R}^n$ *is continuous on* $I \times \Omega$ *and locally Lipschitz on* Ω*, then for every* $(t_0, x) \in I \times \Omega$*, there exists* $\delta > 0$ *such that* $[t_0, t_0 + \delta[\subset I$ *and the Cauchy problem (1.3) admits a unique solution defined on* $[t_0, t_0 + \delta[$*.[5, 2]*

Preuve 3. We apply Peano's theorem.

1.2.1 Maximal Solutions of a Cauchy Problem

In this paragraph we are interested in maximal solutions of problem (1.3).

Recall that a solution $y : \mathbb{J} \to \Omega$ is said to be maximal if it cannot be extended to another solution of the same problem.

Without loss of generality, we assume that $y : [t_0, T) \to \Omega$, where $[t_0, T) \subset I$, is a solution of the Cauchy problem (1.3). We will study the possibility of extending y to another solution $\tilde{y}$ defined on the interval $[t_0, T')$ where $T < T'$ and $[t_0, T') \subset I$. The following lemma will be of great importance in what follows.

Lemma 1.2.9.

Let $f : I \times \Omega \to \mathbb{R}^n$. Then the solution $y : [t_0, T) \to \mathbb{R}^n$ of problem (1.3) is extendable if:

(i) $T < \sup I$.

(ii) There exists $y^* \in \Omega$ such that $\lim_{t \to T^-} y(t) = y^*$.

[5, 2]

Preuve 4.

If the solution y is extendable, then conditions (i) and (ii) are obviously satisfied.

Now suppose that $y : [t_0, T) \to \Omega$ is a solution of problem (1.3) satisfying conditions (i) and (ii). We define an extension of y to the interval $[t_0, T]$ by:

$$\begin{cases} u(t) = y(t), \quad t \in [t_0, T[\\ u(T) = y^*. \end{cases}$$

A simple argument shows that u is a solution of problem (1.3) on the interval $[t_0, T]$. Indeed, it suffices to show that

$$u'(T) = f(T, u(T)) = f(T, y^*).$$

Let $t \in [t_0, T[$ be arbitrary and $h < 0$ sufficiently small (so that $t + h \in [t_0, T[$). Then

$$y(t + h) = y(t) + hy'(t) + o(h) = y(t) + hf(t, y(t)) + o(h).$$

Taking the limit as $t \to T^-$ and using (ii) together with the continuity of f on $I \times \Omega$, we obtain:

$$y(T+h) = y^* + hf(T, y^*) + o(h).$$

It follows that:

$$\frac{u(T+h) - u(T)}{h} = \frac{y^* + hf(T, y^*) + o(h) - y^*}{h} = f(T, y^*) + o(1).$$

Now, letting $h \to 0^-$, we obtain

$$u'(T) = f(T, y^*).$$

By Peano's theorem, the Cauchy problem

$$\begin{cases} y' = f(t, y(t)), \\ y(T) = y^* \end{cases}$$

admits at least one solution v defined on an interval $[T, T') \subset I$. It follows, by the patching principle, that the function $\tilde{y}$ defined by:

$$\begin{cases} \tilde{y}(t) = u(t), & t \in [t_0, T], \\ \tilde{y}(t) = v(t), & t \in [T, T'), \end{cases}$$

is a solution of the Cauchy problem (1.3) defined on $[t_0, T')$. This completes the proof.

Remark 1.2.10.

The previous lemma remains valid if we consider a solution $y :]T, t_0] \to \Omega$, replacing (i) with $T > \inf I$, and in (ii) replacing $t \to T^-$ by $t \to T^+$. In this case, any extension of y will be defined on an interval of the form $]T', t_0]$ where $\inf I < T' < T$.

Remark 1.2.11.

From the previous lemma we deduce that if $y : \mathbb{J} \to \Omega$ is a maximal solution of the Cauchy problem (1.3), and f is continuous on $I \times \Omega$, where I and Ω are open sets, then necessarily the interval $\mathbb{J}$ is open.

The following lemma provides sufficient conditions for condition (ii) of Lemma 1.2.9 to be satisfied.

Proposition 1.2.5.

Let $y : [t_0, T) \to \Omega$ be a solution of problem (1.3) where $T < +\infty$. Suppose there exists a constant $M > 0$ such that:

$$\forall t \in [t_0, T),\ \|f(t, y(t))\| \leq M.$$

Then, there exists $y^* \in \overline{\Omega}$ such that:

$$\lim_{t \to T^-} y(t) = y^*.$$

Preuve 5.

We deduce that:

$$\forall t, s \in [t_0, T),\ \|y(t) - y(s)\| \leq \left| \int_s^t \|f(s, y(s))\| ds \right| \leq M|t - s|.$$

Thus, the Cauchy convergence criterion guarantees that:

$$\lim_{t \to T^-} y(t) \text{ exists.}$$

Theorem 1.2.12.

Every solution $y : \mathbb{J} \to \Omega$ of the Cauchy problem (1.3) can be extended to a maximal solution. [5, 2]

Preuve 6.

Let y be a solution of problem (1.3). Suppose that y is not maximal. Let $\mathcal{S}$ be the set of all solutions extending y. We endow $\mathcal{S}$ with a binary relation, denoted by $\preceq$, defined as follows:

$$\forall x, z \in \mathcal{S},\ x \preceq z \Longleftrightarrow z \text{ extends } x.$$

The set $\mathcal{S}$ is nonempty (since $y \in \mathcal{S}$). Moreover, the set $\mathcal{S}$ endowed with the binary relation $\preceq$ is ordered. Hence, Zorn's lemma allows us to deduce that there exists a maximal element $y^* \in \mathcal{S}$ such that $y \preceq y^*$. Therefore, y^* is maximal. This completes the proof.

Remark 1.2.13.

(i) The previous theorem remains valid even if the function f is not continuous or if one of the sets I or Ω is not open.

(ii) A Cauchy problem may admit several maximal solutions.

We conclude this section with a fundamental result concerning the existence of maximal solutions of a Cauchy problem.

Corollary 1.2.14.

Suppose that I and Ω are open. If $f : I \times \Omega \to \mathbb{R}^n$ is continuous on $I \times \Omega$, then for every $(t_0, x) \in I \times \Omega$, the Cauchy problem (1.3) admits at least one maximal solution defined on an open interval $\mathbb{J}$ containing t_0. [5, 2]

Preuve 7.

The proof follows immediately from Peano's theorem and Theorem 1.2.12.

If the Cauchy problem has the uniqueness property of the solution, for example if f is Lipschitz continuous with respect to the second variable, then the maximal solution, if it exists, is unique. More precisely, we obtain the following result:

Corollary 1.2.15.

Suppose that I and Ω are open. If $f : I \times \Omega \to \mathbb{R}^n$ is continuous on $I \times \Omega$ and locally Lipschitz on Ω, then for every $(t_0, x) \in I \times \Omega$ the Cauchy problem (1.3) admits a unique maximal solution defined on an open interval $\mathbb{J}$ containing t_0.[5, 2]

Preuve 8. The proof follows immediately from the Cauchy–Lipschitz theorem (i.e. Theorem 1.2.8) and Theorem 1.2.12.

Example 1.2.3.

We seek all the maximal solutions of the ODE

$$\begin{cases} y' = y^2, \\ y(0) = 0. \end{cases} \tag{1.6}$$

In order to discuss the nature of the maximal solutions, we must declare the domain of definition of the ODE dynamics. Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and

$$\forall (t, y) \in \mathbb{R} \times \mathbb{R}, \ f(t, y) = y^2.$$

Thus, the (trivial) zero solution is global. On the other hand, we have

$$\frac{df}{dy}(t, y) = 2y,$$

which is continuous on $\mathbb{R}$. We deduce that f is locally Lipschitz with respect to the second variable. Consequently, the zero solution is unique.

Example 1.2.4.

We consider the same previous problem, but with a different initial condition. Let

$$\begin{cases} y' = y^2, \\ y(0) = 1. \end{cases} \tag{1.7}$$

The dynamics are defined as before. Clearly, the zero solution does not satisfy the ODE (1.7). Let us solve the problem explicitly. To this end, we use the method of separation of variables. We obtain:

$$\frac{dy}{y^2} = 1 \Longrightarrow \int \frac{dy}{y^2} = \int 1 \Longrightarrow \frac{-1}{y} = t + C.$$

Hence, every solution of the ODE (1.7) is of the form:

$$y(t) = \frac{-1}{t+C}.$$

Since $y(0) = 1$, we deduce that $C = -1$. Consequently,

$$y(t) = \frac{-1}{t-1}.$$

The solution y is defined on $]-\infty, 1[$. The solution y is maximal. Indeed,

$$\lim_{t \to 1^-} y(t) = -\infty.$$

1.2.2 Global solutions of a Cauchy problem

This subsection is devoted to the study of the existence of solutions of the Cauchy problem (1.3).

Theorem 1.2.16. *Let I be an open interval of $\mathbb{R}$ and let $f : I \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous on $I \times \mathbb{R}^n$. Suppose there exist two functions $h, k : I \to \mathbb{R}$ such that*

$$\forall (t, y) \in I \times \mathbb{R}^n, \ \|f(t, y)\| \leq k(t)\|y\| + h(t).$$

Then, for every $(t_0, x) \in I \times \mathbb{R}^n$, the Cauchy problem (1.3) admits at least one global solution.[5, 2]

Preuve 9.

By virtue of Corollary 1.2.15, the Cauchy problem (1.3) admits at least one maximal solution $y : [t_0, b) \to \Omega$ with $[t_0, b) \subset I$. We deduce that the solution y satisfies

$$\forall t \in [t_0, b), \ \|y(t)\| \leq \|x\| + \int_{t_0}^{t} h(s)\, ds + \int_{t_0}^{t} k(s)\|y(s)\|\, ds.$$

We now show that $b = \sup I$. For contradiction, assume $b \neq \sup I$. Since h and k are continuous on $\mathbb{J}$ and $\mathbb{J}$ is compact, we deduce that there exists a constant $M > 0$ such that

$$\forall t \in [a, b], \ h(t) \leq M \text{ and } k(t) \leq M.$$

Applying Gronwall's Lemma, we therefore obtain

$$\forall t \in [a, b], \ \|y(t)\| \leq \Big(\|x\| + M(b - a)\Big)e^{M(b-a)}.$$

1.2.3 Study of differentiable and integrable functions in normed vector spaces

1.2.4 Normed vector spaces, Banach spaces

An $\mathbb{R}$-vector space X is said to be normed if it can be endowed with a norm $N : X \to \mathbb{R}_+$. We define the following.

Definition 1.2.5. A norm on a vector space X is a mapping $N : X \to \mathbb{R}_+$ satisfying the following three axioms:

- For all $x \in X$, we have $N(x) = 0$ if and only if $x = 0$.
- For all $x \in X$ and $\lambda \in \mathbb{R}$, we have $N(\lambda x) = |\lambda| N(x)$.
- For all $x, y \in X$, we have $N(x + y) \leq N(x) + N(y)$.

Example 1.2.5.

1. The vector space $X = \mathbb{R}^n$ can be equipped with three norms, namely:
    - $N_1(x) = N_1(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} |x_i|$.
    - $N_2(x) = N_2(x_1, x_2, \ldots, x_n) = \left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}}$.
    - $N_\infty(x) = \max\{|x_i| \mid i = 1, 2, \ldots, n\}$.
2. The vector space of bounded real sequences, denoted $l^\infty(\mathbb{R})$, is endowed with the following norm: If $x = (x_n)_n$ is bounded, then

$$N_\infty(x) = \|x\|_\infty = \sup_{n \in \mathbb{N}} |x_n|.$$

Definition 1.2.6. A normed vector space is called a **Banach space** if it is complete.

A topological space is complete if every Cauchy sequence in the space converges.

Example 1.2.6.

1. Let $(E, \|\cdot\|)$ be a Banach space. Let I be an arbitrary set. We denote by $l^{\infty}(I, E)$ the vector space of bounded functions defined on I with values in E. We endow $l^{\infty}(I, E)$ with a norm, denoted by $\|\cdot\|_{\infty}$, defined by:

   $$\forall f \in l^{\infty}(I, E), \quad \|f\|_{\infty} = \sup\{f(x),\ x \in I\}.$$

   It can be shown (as an exercise) that $(l^{\infty}(I, E), \|\cdot\|_{\infty})$ is a Banach space.

2. In the particular case where $I = \mathbb{N}$ and $E = \mathbb{R}$, $l^{\infty}(I, E)$ becomes the vector space of bounded real sequences, and we write:

   $$l^{\infty}(I, E) = l^{\infty}(\mathbb{R}),$$

   which is a Banach space.

1.2.5 Differential of a mapping

Let X and Y be two normed vector spaces over a field $\mathbb{K}$, and let $\Omega \subset X$ be an open set. We consider the function $F : \Omega \to Y$. We denote by $L(X, Y)$ the $\mathbb{K}$-vector space of linear mappings defined on X with values in Y.

Definition 1.2.7. The function f is said to be Fréchet differentiable at $x_0 \in \Omega$ if there exists a linear map denoted $f'(x_0) \in L(X, Y)$ such that:

$$f(x_0 + h) = f(x_0) + f'(x_0)h + \omega(x_0, h), \quad \text{where } \omega(x_0, h) = o(|h|),\ h \to 0.$$

If f is differentiable in the sense of Fréchet at every point $x_0 \in \Omega$, we say that it is Fréchet differentiable on Ω.

Remark 1.2.17.

(i) If the function $f : \Omega \to Y$ is differentiable on X, then we define the derivative function $f' : \Omega \to L(X,Y)$.

If $X = \mathbb{K}$, we can identify $L(\mathbb{K}, Y)$ with Y, and thus $f'(x_0)$ will be identified with a vector.

(ii) If the function f' is continuous, then we say that f is continuously differentiable on Ω and we write $f \in C^1(\Omega)$.

Example 1.2.7. Let the integral operator $F : X \to X$, where $X = C(\mathbb{J})$ and $\mathbb{J} = [a,b]$, be defined by:

$$F(x)(t) = \int_a^b k(t,s) f(s, x(s))\, ds, \quad t \in \mathbb{J}.$$

If $k : \mathbb{J} \times \mathbb{J} \to \mathbb{R}$, f and $\dfrac{\partial f}{\partial x}$ are continuous, then F is differentiable on X and we have:

$$\forall h \in X, \quad (F'(x)(h))(t) = \int_a^b k(t,s) \frac{\partial f}{\partial x}(s, x(s)) h(s)\, ds.$$

Example 1.2.8. Consider the functional $\psi : C(\mathbb{J}) \to \mathbb{R}$ defined by:

$$\psi(x) = \int_a^b \int_0^{x(t)} f(\tau, s)\, ds\, d\tau,$$

where $f : \mathbb{J} \times \mathbb{R} \to \mathbb{R}$ is continuous. Then ψ is continuously differentiable and

$$\psi'(x)(h) = \int_a^b f(\tau, x(\tau)) h(\tau)\, d\tau.$$

Proposition 1.2.6. Let X be a Banach space. If $f : \Omega \to Y$, then:

$$\int_a^b f'(t) = y(b) - y(a) \quad \text{if and only if} \quad y : [a,b] \to X$$

is continuously differentiable.[5, 13]

Proposition 1.2.7. If $f : \Omega \to Y$ is differentiable at $a \in \Omega$, then for all $h \in X$, we have:

$$\lim_{t\to 0} \frac{f(a+th) - f(a)}{t} = df(a) \cdot h.$$

Proposition 1.2.8. If f is differentiable at a, then it is continuous at a.

Remark 1.2.18. We show that f is differentiable at x_0 if and only if there exists a continuous linear mapping, denoted $f'(x_0) \in L(X,Y)$, such that:

$$\lim_{x\to x_0} \frac{f(x) - f(x_0) - f'(x_0)(x - x_0)}{\|x - x_0\|} = 0.$$

Example 1.2.9. Let $X = C^0$ and $I = [0, 1]$. We consider the function $y : I \to X$ defined by:

$$\forall t \in I, \quad y(t) = y_n(t) = \frac{1}{n}\sin(nt).$$

(i) Show that y is Lipschitz. Indeed,

(ii) Show that y is nowhere differentiable.

1.2.6 Solution in the sense of Carathéodory

The existence theorems studied so far require the continuity of the right-hand side defining the ODE. If the continuity condition is relaxed, the existence of a classical local solution (of class C^1) is no longer guaranteed. The following example illustrates this phenomenon.

Example 1.2.10. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by:

$$f(x) = \begin{cases} 1 & \text{if } x < 0, \\ -1 & \text{if } x > 0, \\ -1 & \text{if } x = 0. \end{cases}$$

We consider the Cauchy problem:

$$\begin{cases} y' = f(y(t)), \\ y(0) = 0. \end{cases} \tag{1.8}$$

Suppose that the problem (1.8) admits a C^1 solution y defined to the right of 0, i.e., y is defined at least on an interval $[0, T[$. Then:

$$y'(0) = f(y(0)) = f(0) = -1.$$

Since y' is continuous at 0, we can choose T small enough such that:

$$\forall t \in [0, T[, \qquad y'(t) < 0.$$

It follows that y is strictly decreasing on $[0, T[$. Hence,

$$\forall t \in]0, T[, \qquad y(t) < y(0) = 0.$$

It then follows that:

$$\forall t \in]0, T[, \quad y'(t) = f(y(t)) = 1 \implies y'(0) = 1,$$

which contradicts the fact that we already have $y'(0) = -1$. Finally, we deduce that the Cauchy problem (1.8) does not admit any C^1 solution.

Remark 1.2.19. This phenomenon of non-existence of a C^1 solution for problem (1.8) is, in fact, due to the discontinuity of the function f with respect to the space variable y.

Note here that the problem persists if the function f is discontinuous with respect to the time variable t.

Thus, following the same reasoning as before, the Cauchy problem:

$$\begin{cases} y' = f(t), \\ y(0) = 0. \end{cases} \tag{1.9}$$

admits no classical C^1 solution.

We are therefore led to broaden the class of functions that can be solutions of problem (1.9). In other words, we slightly modify the notion of solution of problem (1.9).

More precisely, we say that $y : \mathbb{J} \to \Omega$ is a solution of problem (1.9) if $y(t) \in \Omega,\ \forall t \in \mathbb{J},\ y(0) = 0,\ y$ is differentiable almost everywhere on $\mathbb{J}$, and

$$y'(t) = f(y(t)) \quad \text{a.e. on } \mathbb{J}.$$

Thus, adopting the previous definition, the function y defined on $\mathbb{R}$ by:

$$y(t) = -|t|$$

becomes a solution of problem (1.9).

It is easy to check that y is defined and continuous on $\mathbb{R}$, and satisfies the relation $y' = f(t)$ almost everywhere on $\mathbb{R}$ (excluding the value 0).

The aim of this section is therefore to define another concept of solution and establish its existence. To this end, a particular class of functions f defining the right-hand side of the ODE will be introduced.

Definition 1.2.8.

A function $f : I \times \Omega \to \mathbb{R}^n$ is said to be of **Carathéodory type** if the following conditions are satisfied:

(i) For almost every $t \in I$, the mapping $x \mapsto f(t, x)$ is continuous on Ω.

(ii) For every $x \in \Omega$, the mapping $t \mapsto f(t, x)$ is Lebesgue-measurable on I.

(iii) For every $(t_0, x) \in I \times \Omega$, there exist $\rho > 0$ and $\delta > 0$ such that

$$[t_0 - \delta,\, t_0 + \delta] \subset I \quad \text{and} \quad B(x, \rho) \subset \Omega.$$

Moreover, there exists a function $h : [t_0 - \delta, t_0 + \delta] \to \mathbb{R}^n$, Lebesgue-integrable, such that

$$\forall (t, y) \in [t_0 - \delta,\, t_0 + \delta] \times B(x, \rho), \quad \|f(t, y)\| \le h(t).$$

Consider now the Cauchy problem:

$$\begin{cases} y'(t) = f(t, y(t)), \\ y(t_0) = x, \end{cases} \tag{1.10}$$

where $f : I \times \Omega \to \mathbb{R}^n$ is of Carathéodory type.

Definition 1.2.9.
A function $y : \mathbb{J} \to \mathbb{R}^n$ is said to be a **solution** of the Cauchy problem (1.10) if:

(i) For all $t \in \mathbb{J},\ y(t) \in \Omega$.

(ii) $y(t_0) = x$, y is absolutely continuous on $\mathbb{J}$, and

$$y'(t) = f(t, y(t)) \quad \text{a.e. on } \mathbb{J}.$$

Theorem 1.2.20.
Suppose $f : I \times \Omega \to \mathbb{R}^n$ is of Carathéodory type. Then, for every $(t_0, x) \in I \times \Omega$, the problem (1.10) admits at least one local solution (in the sense of Definition 1.2.9).[14]

To prove the theorem above, we proceed as follows. Let $(t_0, x) \in I \times \Omega$, and consider the delayed integral equation with parameter $\lambda > 0$, defined by:

$$y_\lambda(t) = \begin{cases} x, & \text{if } t \in [t_0 - \lambda, t_0], \\ x + \displaystyle\int_{t_0}^{t} f(s, y_\lambda(s - \lambda))\, ds, & \text{if } t \in (t_0,\, t_0 + \delta]. \end{cases} \tag{1.11}$$

Method of proof: The idea of the proof is as follows (analogous to Peano's Theorem). First, we show that the integral equation (1.11) admits, for every $\lambda > 0$, at least one solution $y_\lambda : [t_0 - \lambda,\ t_0 + \lambda]$. In contrast to the proof of Peano's theorem, here we prove that the solution of the integral equation is absolutely continuous (the variation of the function equals the primitive of its derivative). Next, by letting $\lambda \to 0$, we show that the sequence $(y_\lambda)_\lambda$ converges uniformly to a function y, where the Arzelà–Ascoli theorem plays a fundamental role. Finally, we prove that the limit y is a solution of the considered Cauchy problem.

To avoid technical complications, we assume $f : I \times \mathbb{R}^n \to \mathbb{R}^n$. We now establish the following lemma:

Lemma 1.2.21.

Let $f : I \times \mathbb{R}^n \to \mathbb{R}^n$ be a Carathéodory function. Suppose there exists $h : I \to \mathbb{R}_+$, locally integrable, such that for almost every $t \in I$:

$$\forall y \in \mathbb{R}^n, \quad \|f(t, y)\| \leq h(t). \tag{1.12}$$

Then, for every $(t_0, x) \in I \times \mathbb{R}^n$ and every $\delta > 0$ with $[t_0, t_0 + \delta] \subset I$, the integral equation (1.11) admits a unique absolutely continuous solution defined on $[t_0 - \lambda,\, t_0 + \delta]$.

Proof. If $y : I \to \mathbb{R}^n$ is continuous on I, then the function $t \mapsto f(t, y(t))$ is measurable on I. From inequality (1.12) we obtain, for almost every $t \in I$,

$$\|f(t, y(t))\| \leq h(t).$$

Thus, $t \mapsto f(t, y(t))$ is locally integrable on I. It follows that the function y_λ is defined and continuous on every interval of the form $[t_0 - \lambda,\, t_0 + i\lambda]$ with $t_0 + i\lambda \leq t_0 + \delta$. Hence $s \mapsto f(s, y_\lambda(s - \lambda))$ is Lebesgue integrable on $[t_0 - \lambda,\, t_0 + (i+1)\lambda]$, for $i = 1, 2, \ldots$. By induction, y_λ is defined on $[t_0 - \lambda,\, t_0 + (i+1)\lambda]$.

On the other hand, y_λ is clearly defined on $[t_0 - \lambda,\, t_0]$. For $t \in [t_0,\, t_0 + \lambda]$, we observe that $s - \lambda \in [t_0 - \lambda, t_0]$, which implies $y_\lambda(s - \lambda) = x$. Hence,

$$y_\lambda(t) = x + \int_{t_0}^{t} f(s, x)\, ds.$$

Therefore y_λ is well-defined on $[t_0, t_0 + \lambda]$. Repeating this argument, y_λ is defined on $[t_0 + \lambda, t_0 + 2\lambda]$, and so forth.

By induction, y_λ is defined on the whole interval $[t_0,\, t_0 + \delta]$. Moreover, y_λ is absolutely continuous (being the primitive of its derivative). The proof is complete. □

We now proceed to another auxiliary result.

Lemma 1.2.22.
Let $f : I \times \mathbb{R}^n \to \mathbb{R}^n$ *be of Carathéodory type, and suppose there exists a function* $h : I \to \mathbb{R}_+$ *satisfying property (1.12). Then, for every* $(t_0, x) \in I \times \mathbb{R}^n$ *and every* $\delta > 0$ *such that* $[t_0, t_0 + \delta] \subset I$*, the Cauchy problem (1.10) admits at least one Carathéodory solution defined on* $[t_0, t_0 + \delta]$*.*

Proof. Let $(t_0, x) \in I \times \mathbb{R}^n$ and $\delta > 0$ with $[t_0, t_0 + \delta] \subset I$. Let $n \in \mathbb{N}$ and consider the delayed integral equation with step $\delta_n = \frac{\delta}{n+1}$:

$$y_n(t) = \begin{cases} x, & t \in [t_0 - \delta_n, t_0], \\ x + \displaystyle\int_{t_0}^{t} f(s, y_n(s - \delta_n))\, ds, & t \in (t_0,\, t_0 + \delta_n]. \end{cases} \tag{1.13}$$

By Lemma 1.2.21, for every $n \in \mathbb{N}$, equation (1.13) admits a unique absolutely continuous solution $y_n : [t_0 - \delta_n,\, t_0 + \delta] \to \mathbb{R}^n$.

Consider the family $\mathcal{F}$ of restrictions of the y_n to $[t_0, t_0 + \delta]$. To prove that $(y_n)_n$ converges uniformly, it suffices to apply the Arzelà–Ascoli theorem. Thus, we need to show that $\mathcal{F}$ is uniformly bounded and equicontinuous.

Uniform boundedness. From (1.12), we have:

$$\forall n \in \mathbb{N},\ \forall t \in [t_0, t_0 + \delta], \quad \|y_n(t)\| \le \|x\| + \int_{t_0}^{t} h(s)\, ds \le \|x\| + \int_{t_0}^{t_0+\delta} h(s)\, ds.$$

Thus $\mathcal{F}$ is uniformly bounded.

Equicontinuity. Again from (1.12):

$$\forall n \in \mathbb{N},\ \forall t, s \in [t_0, t_0 + \delta], \quad \|y_n(t) - y_n(s)\| \le \left| \int_{s}^{t} h(u)\, du \right|.$$

Since h is Lebesgue integrable on $[t_0, t_0 + \delta]$, the map $t \mapsto \int_{t_0}^{t} h(s)\, ds$ is absolutely continuous. Hence $\mathcal{F}$ is equicontinuous.

By Arzelà–Ascoli, there exists a subsequence (y_n) converging uniformly to some function y.

Moreover,

$$\lim_{n\to+\infty} y_n(s-\delta_n) = y(s),$$

uniformly on $[t_0, t_0+\delta]$. Since f is Carathéodory, it follows that for almost every $t \in [t_0, t_0+\delta]$,

$$\lim_{n\to+\infty} f(s, y_n(s-\delta_n)) = f(s, y(s)).$$

By the dominated convergence theorem, passing to the limit yields

$$\forall t \in [t_0, t_0+\delta], \quad y(t) = x + \int_{t_0}^{t} f(s, y(s))\, ds.$$

Thus y is a solution of the Cauchy problem. □

We now proceed to the proof of the main result.

Proof. Let $(t_0, x) \in \Omega$. Since I and Ω are open, there exist $d > 0$ and $r > 0$ such that

$$[t_0 - d,\, t_0 + d] \subset I, \quad B(x, r) = \{\eta \in \mathbb{R}^n : \|\eta - x\| \le r\} \subset \Omega.$$

Define $\rho : \mathbb{R}^n \to \mathbb{R}^n$ by

$$\rho(y) = \begin{cases} y, & y \in B(x, r), \\ \dfrac{r}{\|y - x\|}(y - x) + x, & y \notin B(x, r). \end{cases}$$

The map ρ is continuous and $\rho(\mathbb{R}^n) \subset B(x, r)$. Now define $g : (t_0 - d, t_0 + d) \times \mathbb{R}^n \to \mathbb{R}^n$ by

$$g(t, y) = f(t, \rho(y)).$$

Since f is continuous, the Weierstrass theorem ensures that f is bounded on compact subsets. Thus, for every $d' \in (0, d)$, the Cauchy problem

$$\begin{cases} y' = g(t, y), \\ y(t_0) = x, \end{cases}$$

admits at least one solution $y : [t_0, t_0 + d'] \to \mathbb{R}^n$.

Since $y(t_0) = x$ and y is continuous, for every $r > 0$ there exists $\delta \in (0, d')$ such that

$$\forall t \in [t_0, t_0 + \delta], \quad \|y(t) - x\| \leq r.$$

Thus, $g(t, y(t)) = f(t, y(t))$, and hence y is a solution of the original Cauchy problem. The proof is complete. □

Example 1.2.11.

Consider the Cauchy problem:

$$\begin{cases} y'(t) = y \text{ if } g(t-1), \\ y(0) = 1. \end{cases} \tag{1.14}$$

The Cauchy problem (1.14) has a unique solution $y : \mathbb{R}_+ \to \mathbb{R}$ in the sense of Carathéodory, defined by:

$$\forall t \in \mathbb{R}_+, \; y(t) = e^{|t-1|-1}.$$

In this chapter, we are interested in Cauchy problems of the form:

$$\begin{cases} y' = f(t, y(t)), \\ y(t_0) = x, \end{cases} \tag{1.15}$$

where $f : I \times \Omega \to X,\ X$ is a Banach space, $\Omega \subset X,$ and $I \subset \mathbb{R}$ is an open interval.

The concepts of integrability and differentiability in Banach spaces are detailed in Appendix A.

We begin by defining a solution of problem (1.15).

Definition 1.2.10.
A function $y : \mathbb{J} \to \Omega$ is called a solution of problem (1.15), where $\mathbb{J}$ is a non-empty subinterval of I containing t_0, if:

- $\forall t \in \mathbb{J},\ y(t) \in \Omega$ and $y(t_0) = x$.
- y is differentiable on $\mathbb{J}$ and $\forall t \in \mathbb{J},\ y'(t) = f(t, y(t))$.

Remark 1.2.23. The concept of solutions of ODEs in infinite dimensions is a delicate subject. In infinite dimensions, ODEs rarely admit differentiable solutions.

As in finite dimensions, the following result is a characterization of (classical) solutions of problem (1.15).

Proposition 1.2.9. Suppose that f is continuous on $I \times \Omega$. The function $y : \mathbb{J} \to \Omega$ is a solution of the Cauchy problem (1.15) if and only if y is continuous on $\mathbb{J}$ and

$$\forall t \in \mathbb{J},\ y(t) = x + \int_{t_0}^{t} f(s, y(s))ds.$$

[5]

Preuve 10. We note that

$$\int_{t_0}^{t} f(s, y(s))ds \text{ is in the sense of Bochner.}$$

Remark 1.2.24. The notions of global solution, maximal solution, right solution, left solution, etc. are the same as in finite dimensions.

1.3 Qualitative study of ODEs in infinite dimension (n) (Peano and Cauchy–Lipschitz theorem)

Suppose that $X = \mathbb{R}^n$ and f is continuous on $I \times \Omega$. Then Peano's theorem guarantees that for every $(t_0, x) \in I \times \Omega$, the Cauchy problem (1.15) admits at least one local solution defined on an interval $\mathbb{J}$ containing t_0.

The "naïve" generalization of Peano's theorem to infinite dimension is drastically false.

In 1951, Dieudonné constructed two examples showing that continuity of the dynamics (in our case, the function f) is not sufficient to guarantee the existence of at least one local solution to the Cauchy problem (1.15).

In particular, for every Banach space X of infinite dimension, there exists a Cauchy problem (associated with a continuous function $f : I \times X \to X$) that possesses no local solution.

Thus, to guarantee the existence of at least one local solution, an additional assumption on the function f (redundant in finite dimension) must be imposed.

Definition 1.3.1.

The function f is said to be **compact** if it maps every bounded set in $I \times \Omega$ to a relatively compact set in X.

The theorem below is a variation of Peano's theorem in infinite dimension.

Theorem 1.3.1.

If f is compact and continuous on $I \times \Omega$ where $\Omega \subset \mathbb{R}^n$ is open, then for every $(t_0, x) \in I \times \Omega$, the problem (1.15) admits at least one local solution $y : \mathbb{J} \subset I \to \Omega$ defined on a subinterval $\mathbb{J} \subset I$ containing t_0.

Remark 1.3.2. In finite dimension, the assumption of compactness of f is redundant. Indeed, if f is continuous then it maps every bounded set in $I \times \Omega$ to a bounded set in $\mathbb{R}^n$, which is relatively compact.

Before starting the proof of Theorem 1.3.1, let us first consider the special case where f is continuous on $I \times X$ and $f(I \times X)$ is relatively compact in X. We choose $\lambda > 0$ and $\delta > 0$ such that $[t_0, t_0 + \delta] \subset I$, and consider the delay integral equation:

$$y_\lambda(t) = \begin{cases} x & \text{if } t \in [t_0 - \lambda, t_0], \\ x + \int_{t_0}^{t} f(s, y_\lambda(s - \lambda))ds & \text{if } t \in (t_0, t_0 + \delta]. \end{cases} \tag{1.16}$$

• **Proof method:** First, we show that for every $\lambda > 0$, the integral equation (1.16) admits at least one solution $y_\lambda : [t_0 - \lambda, t_0 + \delta] \to X$.
Next, by letting $\lambda \to 0$, we show that the sequence $(y_\lambda)_\lambda$ converges uniformly to a function y. Finally, we show that the limit y is a solution to the considered Cauchy problem. We note here that the compactness of the function f is essential for the convergence of the approximate solutions to a solution of the considered problem.

Lemma 1.3.3.
If $f : I \times X \to X$ is continuous, then for every $(t_0, x) \in I \times X$, $\lambda > 0$ and $\delta > 0$ such that $[t_0, t_0 + \delta] \subset I$, the delay integral equation (1.16) admits a unique solution defined on $[t_0 - \lambda, t_0 + \delta]$.[5]

Preuve 11.

First, we observe that the function y_λ is uniquely defined on the interval $[t_0 - \lambda, t_0]$ $(y_\lambda \equiv x)$.
Now let $t \in [t_0 - \lambda, t_0)$. It is clear that if $s \in [t_0, t]$ then $s - \lambda \in [t_0 - \lambda, t_0]$, which gives $y_\lambda(s - \lambda) = x$. Consequently,

$$y_\lambda(t) = x + \int_{t_0}^{t} f(s, x)ds,$$

which shows that the function y_λ is also determined on the interval $[t_0-\lambda, t_0]$.
Thus, in a similar manner, we can define y_λ by induction on the intervals
$[t_0+\lambda, t_0+2\lambda],\ [t_0+2\lambda, t_0+3\lambda], \cdots, [t_0+(n-2)\lambda, t_0+(n-1)\lambda]$ such that $n\lambda \geq t_0+\delta$.
Therefore, the function y_λ is defined on the interval $[t_0, t_0+\delta]$ and is continuous. The proof is thus complete.

We now move on to another auxiliary existence result.

Lemma 1.3.4.
Let $f : I \times X \to X$ be continuous and suppose $f(I\times X)$ is relatively compact. Then, for every $(t_0, x) \in I \times X$ and $\delta > 0$ such that $[t_0, t_0+\delta] \subset I$, the delay equation (1.16) has a unique solution defined on $[t_0, t_0+\delta]$.[13]

Preuve 12.

Let $(t_0, x) \in I \times X$ and $\delta > 0$ such that $[t_0, t_0+\delta] \subset I$. Let $n \in \mathbb{N}$. Consider the delayed integral equation with $\delta_n = \dfrac{1}{n+1}$:

$$y_n(t) = \begin{cases} x & \text{if } t \in [t_0-\delta_n, t_0] \\ x + \int_{t_0}^{t} f(s, y_n(s-\delta_n))ds & \text{if } t \in (t_0, t_0+\delta_n]. \end{cases} \tag{1.17}$$

By virtue of Lemma 1.3.3, for every $n \in \mathbb{N}$, the delayed integral equation (1.17) admits a unique solution $y_n : [t_0-\delta_n, t_0+\delta] \to X$. In the sequel, we consider the family
$\mathcal{F} = \{y_n,\ n \in \mathbb{N}\}$ consisting of the restrictions of y_n to the interval $[t_0, t_0+\delta]$. We now aim to prove that the family $\mathcal{F}$ is relatively compact in the space $C([t_0, t_0+\delta], X)$. This amounts to showing that $\mathcal{F}$ satisfies the assumptions of the Arzelà–Ascoli theorem, i.e., $\mathcal{F}$ is uniformly bounded and equicontinuous.
The family $\mathcal{F}(t) = \{y_n(t),\ n \in \mathbb{N}\}$ is relatively compact: We need to prove that for every $t \in [t_0, t_0+\delta]$, the set

$$\mathcal{F}(t) = \{y_n(t),\ n \in \mathbb{N}\}$$

is relatively compact in X. This follows directly from the assumption that $f(I \times X)$ is relatively compact in X. Therefore, we deduce that the set

$$\mathcal{F}(t) = \{y_n(t),\ n \in \mathbb{N}\} = \left\{x + \int_{t_0}^{t} f(s, y_n(s-\delta_n))ds,\ n \in \mathbb{N}\right\}$$

is relatively compact in X for all $t \in [t_0, t_0 + \delta)$.

The family $\mathcal{F}$ is equicontinuous: The set $\mathcal{F}$ is equicontinuous. Indeed,

$$\forall (t,s) \in [t_0, t_0+\delta),\ \|y_n(t) - y_n(s)\| \leq \left| \int_{s}^{t} \|f(\sigma, y_n(\sigma - \delta_n))\| d\sigma \right| \leq M|t-s|,$$

for some constant M and for all $n \in \mathbb{N}$. Consequently, $\mathcal{F}$ is equicontinuous on $[t_0, t_0 + \delta]$. By applying the Arzelà–Ascoli theorem, we deduce that $(y_n)_n$ converges uniformly to a function $y : [t_0, t_0 + \delta] \to X$.

On the other hand, we have:

$$\lim_{n} y_n(s - \delta_n) = y(s)$$

uniformly on $[t_0, t_0 + \delta]$. Since f is continuous on $I \times X$, by passing to the limit in (1.17), we deduce that

$$\forall t \in [t_0, t_0 + \delta],\ y(t) = x + \int_{t_0}^{t} f(s, y(s))ds.$$

It follows that $y : [t_0, t_0 + \delta] \to X$ is a solution to the Cauchy problem (1.15).

We now proceed to the proof of the main theorem of this section (Theorem 1.3.1).

Preuve 13.

Let $(t_0, x) \in \Omega$. Since Ω is open, there exist $d > 0$ and $r > 0$ such that

$$[t_0 - d, t_0 + d] \times B(x, r) \subset D \text{ and } B(x,r) = \{\eta \in X,\ \|\eta - x\| \leq r\}.$$

Define the function $\rho : X \to X$ by:

$$\rho(y) = \begin{cases} y & \text{if } y \in B(x,r) \\ \dfrac{r}{\|y - x\|}(y - x) + x & \text{if } y \in X - B(x,r). \end{cases}$$

Hence, ρ is continuous and $\rho(X) \subset B(x,r)$. Now define $g : (t_0 - d, t_0 + d) \times X \to X$ by:

$$\forall (t,y) \in (t_0 - d, t_0 + d) \times X,\ \ g(t,y) = f(t, \rho(y)).$$

The set $f((t_0 - d, t_0 + d) \times B(x,r))$ is relatively compact since f is b-compact. Therefore, g is continuous and $g((t_0 - d, t_0 + d) \times X)$ is relatively compact. By virtue of Lemma 1.3.3, we deduce that for every $d' \in (0,d)$, the Cauchy problem

$$\begin{cases} y' = g(t,y) \\ y(t_0) = x, \end{cases}$$

admits at least one solution $y : [t_0, t_0 + d'] \to X$.
Since y is continuous at t_0 and $y(t_0) = x$, we deduce that for any $r > 0$, there exists $\delta \in (0, d')$ such that for all $t \in [t_0, t_0 + \delta]$, we have $\|y(t) - x\| \leq r$. Thus,

$$g(t, y(t)) = f(t, y(t)).$$

Consequently, the function y is a solution to the Cauchy problem (1.15). The proof is complete.

Remark 1.3.5.
We note here that the missing argument in Peano's result is the impossibility of making the local approximate solutions converge by means of Ascoli's theorem. The latter itself fails because the closed balls of X are not compact: by Riesz's theorem, the closed unit ball of an infinite-dimensional normed space is never compact. More precisely, one can prove that in any separable infinite-dimensional Banach space X there exists a continuous function $f : X \to X$ such that the ODE $y' = f(y)$ admits no local solution.

Remark 1.3.6.

The results established in finite dimensions concerning the existence of maximal solutions remain valid in infinite dimensions. More precisely, we have the above variation of Peano's theorem concerning the existence of a maximal solution of a Cauchy problem in infinite dimension.

Theorem 1.3.7.

If f is compact and continuous on $I \times \Omega$, where $\Omega \subset \mathbb{R}^n$ is open, then for every $(t_0, x) \in I \times \Omega$, problem (1.9) admits at least one maximal solution $y : \mathbb{J} \subset I \to \Omega$ defined on an open interval $\mathbb{J} \subset I$ containing t_0. [13]

1.3.1 Cauchy Problem and Integral Equation

We are interested in providing a result of existence and uniqueness of the solution of a Cauchy problem in infinite dimension.

Remark 1.3.8.

The proof of the Cauchy–Lipschitz theorem is analogous to that established in finite dimensions.

In what follows, we will present and prove it in a different way, based on Picard's approximations. Broadly speaking, the method of proving the existence of a solution in Banach spaces (including in finite dimension) proceeds in three steps:

- Construct a sequence of approximate solutions of the considered Cauchy problem.
- Show that the sequence built in the first step converges uniformly.
- Finally, prove that the limit obtained in the second step is indeed a solution of the Cauchy problem.

We now collect the following assumptions on the function f:

(H1) The function $f : R_0 = [t_0, t_0 + \delta] \times B(x, r) \to X$ is continuous and

$$\exists M_0 > 0,\ \forall (t, y) \in R_0,\ \|f(t, y)\| \leq M_0.$$

(H2) There exists a function $g : [t_0, t_0 + a] \times [0, 2b] \to \mathbb{R}_+$ such that

$$\forall (t, u) \in [t_0, t_0 + a] \times [0, 2b],\ 0 \leq g(t, u) \leq M_1.$$

Moreover, $g(t, 0) \equiv 0$, for all $t \in [t_0, t_0 + \delta]$ the function $g(t, \cdot)$ is strictly increasing, and the zero function is the unique solution of the Cauchy problem

$$u' = g(t, u),\ u(t_0) = 0. \tag{1.18}$$

(H3) f satisfies the following relation (an extension of the Lipschitz condition):

$$\forall (t, y), (t, z) \in R_0,\ \|f(t, y) - f(t, z)\| \leq g(t, \|y - z\|).$$

The above result (Cauchy–Lipschitz theorem in infinite dimension) extends in a natural way the one already established in finite dimension. More precisely, we have the following theorem:

Theorem 1.3.9.

Suppose that assumptions $(H1) \sim (H2)$ *are satisfied. Then the successive Picard approximations (in infinite dimension) defined by:*

$$y_{n+1}(t) = x + \int_{t_0}^{t} f(s, y_n(s))\, ds, \quad n = 0, 1, 2, \cdots \tag{1.19}$$

exist on $[t_0, t_0+\alpha]$*, where* $\alpha = \min\{a, \frac{b}{M}\}$ *and* $M = \max\{M_0, M_1\}$*, and converge uniformly to a function* y*, which is the unique solution of the Cauchy problem (1.9). [15]*

! • **Existence of the solution:** We prove by induction that the successive approximations (1.19) are defined and continuous on !b α

Moreover, we have:

$$\forall t \in [t_0, t_0 + \alpha], \ \|y_{n+1}(t) - x\| \leq b, \quad n = 0, 1, 2, \cdots .$$

We now consider the following approximations:

$$\begin{cases} u_0(t) = M(t - t_0), \\ \\ u_{n+1}(t) = \int_{t_0}^{t} g(s, u_n(s))\, ds, \quad t \in [t_0, t_0 + \alpha]. \end{cases}$$

Again, by induction, we show that the successive approximations above are well defined, and we have:

$$0 \leq u_{n+1}(t) \leq u_n(t), \quad t \in [t_0, t_0 + \alpha].$$

By the Ascoli–Arzelà theorem, we deduce that the sequence (u_n) converges uniformly to a function u on $[t_0, t_0 + \alpha]$. From assumption (ii), we deduce that $u \equiv 0$ on $[t_0, t_0 + \alpha]$.

On the other hand, we show by induction that

$$\forall t \in [t_0, t_0 + \alpha], \ \|y_{n+1}(t) - y_n(t)\| \leq u_n(t), \quad n = 0, 1, 2, \cdots .$$

As a consequence,

$$\forall t \in [t_0, t_0 + \alpha],\ \|y'_{n+1}(t) - y'_n(t)\| \leq g(t, u_{n-1}(t)).$$

Now, let $n \leq m$. We then have:

$$\forall t \in [t_0, t_0 + \alpha],\ \|y'_{n+1}(t) - y'_n(t)\| \leq g(t, u_{n-1}(t)) + g(t, u_{m-1}(t)) + g(t, \|y_n(t) - y_m(t)\|).$$

Since $u_{n+1}(t) \leq u_n(t)$ on $[t_0, t_0 + \alpha]$, it follows that:

$$\frac{d}{dt}\Big(\|y_n(t) - y_m(t)\|\Big) \leq g(t, \|y_n(t) - y_m(t)\|) + 2g(t, u_{n-1}(t)).$$

Consequently,

$$\forall t \in [t_0, t_0 + \alpha],\ \|y_n(t) - y_m(t)\| \leq r_n(t),$$

where $r_n(t)$ is the maximal solution of the Cauchy problem:

$$v' = g(t, v) + 2g(t, u_{n-1}(t)), \quad v_n(t_0) = 0, \quad n = 0, 1, \cdots .$$

But $2g(t, u_{n-1}(t))$ tends uniformly to 0 on $[t_0, t_0 + \alpha]$ as $n \to +\infty$. This implies that y_n converges uniformly to y as $n \to +\infty$. Finally, we show that y is a solution of the Cauchy problem.

- **Uniqueness of the solution:** To show that the solution y is unique, suppose that z is another solution of the Cauchy problem. Define

$$\forall t \in [t_0, t_0 + \alpha],\ m(t) = \|y(t) - z(t)\|.$$

Thus, $m(t_0) = 0$. Moreover, we have:

$$\forall t \in [t_0, t_0 + \alpha],\ m'(t) \leq \|y(t) - z(t)\| \leq g(t, m(t)).$$

From assumption (iii), we deduce that:

$$\forall t \in [t_0, t_0 + \alpha],\ m(t) \leq r(t),$$

where r is the unique maximal solution of (1.18). From assumption **(H2)** we obtain $r \equiv 0$, which implies that $y \equiv z$. The proof is thus complete.

Remark 1.3.10.

We note here that if assumption (H3) is weakened, the result may fail.

1.3.2 Solved Exercises

Exercise 1:

Let F be a function of class C^2 on $\mathbb{R}^n$ with values in $\mathbb{R}$ such that

$$\lim_{\|x\|\to+\infty} F(x) = +\infty,$$

and let $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$. Consider the Cauchy problem

$$\begin{cases} \dfrac{d}{dt}x(t) &= -\nabla F(x(t)), \quad t \in \mathbb{R}, \\ x(t_0) &= x_0, \quad x_0 \in \mathbb{R}^n. \end{cases} \tag{1.20}$$

1. Show that (1.20) admits a unique maximal solution defined on an interval $]T_-, T_+[\subset \mathbb{R}$.

Solution:

1. Since F is of class C^2, we have

   $$\nabla F(x) = \left(\frac{\partial F}{\partial x_j}(x)\right)_{j=1,\dots,n},$$

   which is of class C^1. Moreover, $\Omega = \mathbb{R}^n$ is convex, hence ∇F is locally Lipschitz continuous in x, uniformly with respect to t. By the **Cauchy–Lipschitz theorem**, the problem (1.20) admits a unique maximal local solution defined on an interval $J =]T^-, T^+[\subset \mathbb{R}$.

2. For all $t \in J$ and for any solution $x(t) = (x_1(t), \dots, x_n(t))$ of (1.20), we have

   $$\frac{d}{dt}F(x(t)) = \sum_{j=1}^{n} \frac{\partial F(x(t))}{\partial x_j}\frac{dx_j(t)}{dt}.$$