Sheet 2

Exercise 01.
Show that a norm is convex.

Exercise 02.
Show that the indicator function of a set Ω defined by:

$$1_\Omega(x) = \begin{cases} 0 & \text{if } x \in \Omega \\ +\infty & \text{otherwise} \end{cases}$$

is convex if and only if O is convex.

Exercise 03.
Let U be a convex subset of a vector space V. Show that $f : U \subset V \to \mathbb{R}$ is convex if and only if the set:

$$\text{epi}(f) = \{(v, a) \in U \times \mathbb{R} \mid a \geq f(v)\}$$

is a convex subset of $U \times \mathbb{R}$.

Exercise 04.
Let F be a function from $\mathbb{R}^n$ to $\mathbb{R}$. Define the function $\phi : \mathbb{R}^n \to \mathbb{R}$ by:

$$\phi(\alpha) = \frac{F(u + \alpha v) - F(v)}{\alpha}, \quad \forall \alpha > 0, \forall (u, v) \in \mathbb{R}^n \times \mathbb{R}^n.$$

Show that if F is convex then ϕ is increasing.

Exercise 05.
Let $\{f_i\}_{i \in I}$ be any family of convex functions $f_i : U \subset V \to \mathbb{R}$. Prove that the function $\sup_{x \in R^n} f_i$ is convex.

Exercise 06.
Prove Young's inequality: For all $a, b > 0$, and for all $p, q \in \mathbb{N}$ such that $\frac{1}{p} + \frac{1}{q} = 1$.
$ab \leq \dfrac{1}{p} a^p + \dfrac{1}{q} b^q$

Exercise 07.
Let f be a convex function from $\mathbb{R}^n$ to $\mathbb{R}$. Show that:

$$\forall (\lambda_i)_{i=1}^n, (x_i)_{i=1}^n \in (\mathbb{R}^n)^n \text{ such that } \sum_{i=1}^n \lambda_i = 1, f\left(\sum_{i=1}^n \lambda_i x_i\right) \leq \sum_{i=1}^n \lambda_i f(x_i).$$

Exercise 08. (Characterization of convexity)
Let $\Omega \subset \mathbb{R}^n$ be an open set, $U \subset \Omega$ convex, and $f : \Omega \to \mathbb{R}$ a C^1 function. Then the following are equivalent:

1. f is convex on U.
2. $f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle$, $\forall x, y \in U$.
3. ∇f is monotone on U.

Exercise 09.
Let f be C^2 on U with U convex. Then f is convex on U if and only if:

$$\langle \nabla^2 f(x)(y - x), y - x \rangle \geq 0, \quad \forall x, y \in U.$$

Exercise 10.

Let f be the real-valued function defined by:

$$\forall x \in \mathbb{R}, \quad f(x) = \sqrt{x^4+1} - x + 3.$$

Show that f admits a global minimum on $\mathbb{R}$.

Exercise 11.
Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by:

$$\forall x \in \mathbb{R}, \quad f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-\theta)^2}{2}}.$$

We fix $x_1, \ldots, x_n \in \mathbb{R}$. Solve the following problem:

$$\max_{\theta \in \mathbb{R}} g(\theta) = \prod_{i=1}^{n} f(x_i).$$

Exercise 12
We consider the function

$$f(x) = \tfrac{1}{2} x^T A x - b^T x + c,$$

where A is a real symmetric matrix.

1. Compute $\nabla f(x)$ and $\nabla^2 f(x)$.
2. Let $\lambda_{\min}, \lambda_{\max}$ be the smallest and largest eigenvalues of A. Show that

$$\forall x \in \mathbb{R}^n, \quad \lambda_{\min}\|x\|^2 \leq \langle Ax, x\rangle \leq \lambda_{\max}\|x\|^2.$$

3. Suppose that A is positive definite:
   (a) Show that f is coercive.
   (b) Deduce that f admits a (unique) global minimum.
   (c) Show that x^* is a solution of the system $Ax = b$ if and only if x^* achieves the (unique) minimum of $f(x)$ on $\mathbb{R}^n$.
4. If A is not positive semidefinite, show that f does not admit a minimum on $\mathbb{R}^n$.
5. **Application:** Let

$$f : \mathbb{R}^3 \to \mathbb{R}, \quad f(x,y,z) = x^2 + y^2 + z^2 + xy + yz + xz - 3x - 4y + 4.$$

   (a) Express f in the form of a quadratic function

$$f(x) = \tfrac{1}{2} x^T A x - b^T x + c,$$

   where A is a symmetric matrix, $b \in \mathbb{R}^3$, and $c \in \mathbb{R}$.
   (b) Solve the minimization problem

$$\min_{x \in \mathbb{R}^3} f(x).$$

Sheet 02 (Solutions)

Exercise 01.
Solution. Let $\|\cdot\|$ be a norm. To prove it is convex, we show that:

$$\|\lambda x + (1-\lambda)y\| \leq \lambda\|x\| + (1-\lambda)\|y\|, \quad \forall x, y, \lambda \in [0,1].$$

Using triangle inequality and homogeneity,

$$\|\lambda x + (1-\lambda)y\| \leq \|\lambda x\| + \|(1-\lambda)y\| = \lambda\|x\| + (1-\lambda)\|y\|.$$

Thus, a norm is convex.

Exercise 02.
Solution. If Ω is convex, then for $x, y \in \Omega$ and $\lambda \in [0,1]$, $\lambda x + (1-\lambda)y \in \Omega$, thus

$$1_\Omega(\lambda x + (1-\lambda)y) = 0 \leq \lambda 1_\Omega(x) + (1-\lambda)1_\Omega(y) = 0.$$

If Ω is not convex, take $x, y \in \Omega$ but $\lambda x + (1-\lambda)y \notin O$, then 1_Ω fails convexity. Hence, 1_Ω is convex iff Ω is convex.

Exercise 03.
Solution. If f is convex, for any $(v_1, a_1), (v_2, a_2) \in epi(f)$ and $\lambda \in [0,1]$,

$$\lambda a_1 + (1-\lambda)a_2 \geq \lambda f(v_1) + (1-\lambda)f(v_2) \geq f(\lambda v_1 + (1-\lambda)v_2).$$

Hence $(\lambda v_1 + (1-\lambda)v_2, \lambda a_1 + (1-\lambda)a_2) \in epi(f)$. The converse follows similarly.

Exercise 04.
Solution. Given F convex, consider

$$\phi(\alpha) = \frac{F(u + \alpha v) - F(v)}{\alpha}.$$

Since the secant slope of a convex function is increasing, ϕ is increasing in α.

Exercise 05.
Solution. Let $f(x) = \sup_i f_i(x)$. For x, y and $\lambda \in [0,1]$,

$$\begin{aligned}
&f(\lambda x + (1-\lambda)y) = \sup_i f_i(\lambda x + (1-\lambda)y) \\
&\leq \sup_i [\lambda f_i(x) + (1-\lambda)f_i(y)] \\
&\leq \lambda \sup_i f_i(x) + (1-\lambda)\sup_i f_i(y) = \lambda f(x) + (1-\lambda)f(y).
\end{aligned}$$

Hence f is convex.

Exercise 06.
Solution. Young's inequality:

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \quad \text{with } \frac{1}{p} + \frac{1}{q} = 1.$$

Follows from convexity of $x \mapsto x^p$ and its conjugate.

Exercise 07.
Solution. By Jensen's inequality for convex f,

$$f\left(\sum \lambda_i x_i\right) \leq \sum \lambda_i f(x_i).$$

Exercise 08.

Solution. (1)⇒(2): First order characterization of convexity. (2)⇒(3): Differentiating shows monotonicity. (3)⇒(1): Integration along segment yields convexity.

Exercise 09.

Solution. f convex iff Hessian is positive semi-definite:

$$\langle \nabla^2 f(x)(y-x), y-x\rangle \geq 0, \forall x, y.$$

Exercise 10.

We have

$$\lim_{|x|\to+\infty} f(x) = \lim_{|x|\to+\infty} \left(\sqrt{x^4+1} - x + 3\right) = +\infty.$$

The function $f(x)$ is coercive and continuous. Therefore, it admits at least one global minimum point on $\mathbb{R}$.

Exercise 11.

First, we write

$$g(\theta) = \left(\frac{1}{\sqrt{2\pi}}\right)^n \exp\left(-\frac{(x_1-\theta)^2}{2} - \cdots - \frac{(x_n-\theta)^2}{2}\right).$$

Then, the derivative is

$$g'(\theta) = \left(\sum_{i=1}^n x_i - n\theta\right) g(\theta).$$

Since $g(\theta) > 0$, we have

$$g'(\theta) = 0 \iff \theta^* = \frac{1}{n}\sum_{i=1}^n x_i.$$

Now we compute the second derivative:

$$g''(\theta) = \left(\sum_{i=1}^n x_i - n\theta\right) g'(\theta) - n g(\theta).$$

In particular, at the critical point θ^*:

$$g''(\theta^*) = -n g(\theta^*) < 0.$$

Therefore,

$$\boxed{\theta^* = \frac{1}{n}\sum_{i=1}^n x_i}$$

is the unique global maximizer of g.

Exercise 12.

We consider the function

$$f(x) = \tfrac{1}{2}x^T Ax - b^T x + c,$$

with A a real symmetric matrix.

We have

$$\nabla f(x) = Ax - b, \quad \nabla^2 f(x) = A.$$

Let $x \in \mathbb{R}^n$. Since A is symmetric, there exists an orthonormal eigenbasis $\{v_i\}_{i=1}^n$ of $\mathbb{R}^n$ with eigenvalues $(\lambda_i)_{i=1}^n$. We can write

$$x = \sum_{i=1}^n \beta_i v_i.$$

Then

$$\langle Ax, x\rangle = \left\langle A\sum_{i=1}^n \beta_i v_i, \sum_{i=1}^n \beta_i v_i \right\rangle = \sum_{i=1}^n \beta_i^2 \lambda_i.$$

Hence

$$\lambda_{\min} \sum_{i=1}^n \beta_i^2 \leq \sum_{i=1}^n \beta_i^2 \lambda_i \leq \lambda_{\max} \sum_{i=1}^n \beta_i^2,$$

which yields

$$\lambda_{\min}\|x\|^2 \leq \langle Ax, x\rangle \leq \lambda_{\max}\|x\|^2.$$

Since $A \succ 0$, we have $\lambda_{\min} > 0$. Then

$$\langle Ax, x\rangle \geq \lambda_{\min}\|x\|^2.$$

Thus

$$f(x) = \tfrac{1}{2}x^T Ax - b^T x + c \geq \tfrac{\lambda_{\min}}{2}\|x\|^2 - \|b\|\|x\| + c.$$

As $\|x\| \to +\infty$, the quadratic term dominates, and

$$\lim_{\|x\|\to+\infty} f(x) = +\infty.$$

Therefore, f is coercive and continuous, hence it admits at least one global minimum. Since A is positive definite, f is strictly convex, which implies the minimum is unique.

Moreover, x^* solves $Ax = b$ if and only if

$$\nabla f(x^*) = Ax^* - b = 0,$$

i.e. x^* is exactly the unique global minimizer of f.

Since $\nabla^2 f(x) = A$ is constant, if A is not positive semidefinite, the quadratic form $\frac{1}{2}x^T Ax$ is not bounded below. Hence f does not admit a global minimum on $\mathbb{R}^n$.

We consider

$$f(x, y, z) = x^2 + y^2 + z^2 + xy + yz + xz - 3x - 4y + 4.$$

We compute

$$\nabla f(x, y, z) = \begin{pmatrix} 2x + y + z - 3 \\ x + 2y + z - 4 \\ x + y + 2z \end{pmatrix}, \qquad f(0, 0, 0) = 4.$$

Thus

$$b = \nabla f(0,0,0) = \begin{pmatrix} -3 \\ -4 \\ 0 \end{pmatrix}, \qquad c = 4,$$

and

$$A = \nabla^2 f(x,y,z) = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

Therefore,

$$f(x,y,z) = \tfrac{1}{2}\,(x,y,z)A(x,y,z)^T - b^T(x,y,z)^T + c.$$

By Sylvester's criterion:

$$\det(2) = 2 > 0, \quad \det\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = 3 > 0, \quad \det(A) = 4 > 0.$$

Thus A is positive definite. Therefore f admits a unique minimizer $x^* \in \mathbb{R}^3$, which is the unique solution of $Ax = b$:

$$\begin{cases} 2x + y + z = -3, \\ x + 2y + z = -4, \\ x + y + 2z = 0. \end{cases}$$

Solving this system yields

$$(x,y,z) = \left(-\tfrac{5}{4},\ -\tfrac{9}{4},\ \tfrac{7}{4}\right).$$