

**Abbreviations:**

**LP: Linear Programming | NLP: Nonlinear Programming | QP: Quadratic Programming**  
**MILP: Mixed-Integer Linear Programming | MINLP: Mixed-Integer Nonlinear Programming**  
**SOCP: Second-Order Cone Programming | SDP: Semidefinite Programming**  
**B&B: Branch-and-Bound | B&C: Branch-and-Cut | SA: Simulated Annealing**  
**GA: Genetic Algorithms | PSO: Particle Swarm Optimization | ACO: Ant Colony Optimization**  
**DE: Differential Evolution | RL: Reinforcement Learning**

Figure 1: Comprehensive diagram of the main types of optimization and their relationships, with abbreviations explained below the figure.

# Chapter 2

## Unconstrained Minimization

### Introduction

Unconstrained minimization refers to the optimization of an objective function without the presence of explicit constraints on the decision variables. In such problems, the goal is to find a point where the function attains its minimum value over its entire domain. This class of optimization problems forms the foundation of numerical optimization, with applications in data fitting, machine learning, economics, and engineering design. Analytical and numerical methods, such as gradient-based algorithms and Newton-type methods, are widely used to solve unconstrained minimization problems efficiently.

Let  $f : K \rightarrow \mathbb{R}$ . The **unconstrained minimization problem** is defined as:

$$\min_{x \in \mathbb{R}^n} f(x) \tag{2.1}$$

We aim to solve:

$$\min_{x \in K \subset \mathbb{R}^n} f(x) \quad \text{or} \quad \max_{x \in K \subset \mathbb{R}^n} f(x),$$

that is, we seek  $v$ , the optimal value, and  $x^*$  such that:

$$f(x^*) = v.$$

The local and global minima of  $f$  on  $\mathbb{R}^n$  are defined as follows:

### 2.1 Definitions

**Definition 2.1.**

We say that  $x^*$  is a **local minimum** of  $f$  if:

$$\exists V \in \mathcal{V}(x_0) \text{ such that } \forall x \in V, f(x) \geq f(x^*).$$

**Definition 2.2.**

We say that  $x^*$  is a **strict local minimum** of  $f$  if:

$$\exists V \in \mathcal{V}(x_0) \text{ such that } \forall x \in V, f(x) > f(x^*).$$

**Definition 2.3.**

We say that  $x^*$  is a **global minimum** if:

$$\forall x \in \mathbb{R}^n, f(x) \geq f(x^*).$$

**Definition 2.4.**

We say that  $x^*$  is a **strict global minimum** if:

$$\forall x \in \mathbb{R}^n, f(x) > f(x^*).$$

**Definition 2.5.**

A sequence  $(x_n)_{n \in \mathbb{N}}$  of points in  $\mathbb{R}^n$  is called a **minimizing sequence** if:

$$\lim_{n \rightarrow +\infty} f(x_n) = \inf_{x \in \mathbb{R}^n} f(x).$$

**Definition 2.6.**

A stationary point that is neither a minimum nor a maximum is called a **singular point**.

**Definition 2.0.7.**

$x_0$  is a **stationary point** if and only if  $\nabla f(x_0) = 0$ .

**Remark 2.1.**

1. A global minimum is clearly a local minimum.
2. If we simply say “minimum”, it is understood as a global minimum.
3. If we know all local minima, then the smallest one is the global minimum.
4. Any isolated optimal solution is a strict local optimal solution. The converse is not always true.

**Definition: Lower Semicontinuity[11]**

Let  $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$  be a function. We say that  $f$  is **lower semicontinuous** at a point  $x_0 \in \mathbb{R}^n$  if:

$$\liminf_{x \rightarrow x_0} f(x) \geq f(x_0).$$

Equivalently, for any sequence  $(x_k)$  converging to  $x_0$ :

$$f(x_0) \leq \liminf_{k \rightarrow \infty} f(x_k).$$

## 2.2 Existence and Uniqueness of the Problem's Solution

Let us now examine the questions of existence and uniqueness of the solution to problem (1). In finite dimension, the uniqueness of a potential solution is generally established independently of its existence, most often based on the convexity of the set  $K$  and the strict convexity of the functional  $f$ .

### 2.1.1 In practice

We denote problem (2.2) as:

$$\min_{x \in K} f(x) \tag{2.2}$$

and problem (2.3) as:

$$\min_{x \in \mathbb{R}^n} f(x). \tag{2.3}$$

**Lemma 2.1.1.** Let  $K \subset \mathbb{R}^n$  be an open set. Then,

$$x^* \text{ is a solution to (1)} \Leftrightarrow \begin{cases} x^* \text{ is a solution to (2.1)} \\ x^* \in K. \end{cases}$$

In practice, to solve (1), we first solve (2.1), which is simpler, then:

1. If (2.1) has a solution  $x^* \in K$ , then  $x^*$  is a solution to (1). Otherwise, if  $x^* \notin K$ , then (1) has no solution.
2. If (2.1) has no solution, we cannot conclude anything about (1).

**Definition 2.2.1** (Strong convexity). A differentiable function  $f : K \rightarrow \mathbb{R}$ , with  $K \subseteq E$  convex, is said to be  $\alpha$ -strongly convex for some  $\alpha > 0$  if for all  $x, y \in K$ :

$$f(y) \geq f(x) + \nabla f(x) \cdot (y - x) + \frac{\alpha}{2} \|y - x\|^2.$$

## 2.3 Existence of a Solution

**Theorem 2.1.1 (Existence : Weierstrass Theorem)[11]**

Let  $K$  be a non-empty compact (closed and bounded) subset of  $\mathbb{R}^n$ , and let  $f : K \rightarrow \mathbb{R}$  be a continuous function on  $K$ . Then  $f$  admits at least one minimum  $x^*$  on  $K$ . In other words,

$$\exists x^* \in K \text{ such that } \forall x \in \mathbb{R}^n, f(x) \geq f(x^*).$$

**Proof.**

Let  $(x_n)$  be a minimizing sequence of  $f$  on  $K$  such that

$$x_n \in K, \forall n \in \mathbb{N}, \quad \text{and} \quad \lim_{n \rightarrow +\infty} f(x_n) = \inf_{x \in K} f(x).$$

Since  $K$  is bounded,  $(x_n)$  is bounded, so we can extract a subsequence converging to  $x^* \in K$  (as  $K$  is closed). By continuity,

$$\lim_{n \rightarrow +\infty} f(x_n) = f(x^*),$$

thus,

$$f(x^*) = \inf_{x \in K} f(x).$$

Therefore,  $f$  attains its minimum on  $K$ .

**Theorem 2.1.2.**[11]

Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be continuous and coercive. Then  $f$  admits at least one minimum on  $\mathbb{R}^n$ .

**Proof.**

Let  $(x_n)$  be a minimizing sequence in  $\mathbb{R}^n$ , then

$$\lim_{n \rightarrow +\infty} f(x_n) = f(x^*),$$

thus,

$$f(x^*) = \inf_{x \in \mathbb{R}^n} f(x).$$

Hence,  $f$  attains its minimum on  $\mathbb{R}^n$ .

**Theorem 2.1.3 (Uniqueness).**[11]

If  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is strictly convex, then  $f$  admits a unique minimum  $x^*$  such that

$$\forall x \in \mathbb{R}^n, f(x) \geq f(x^*).$$

**Proof.**

Assume  $f$  is strictly convex and suppose there exist two distinct minima  $x^*, x'^*$  with

$$f(x^*) = f(x'^*) = \inf_{x \in \mathbb{R}^n} f(x).$$

Let  $x''^* = \lambda x'^* + (1 - \lambda)x^*$ , with  $\lambda \in (0, 1)$ . Then

$$f(x''^*) < \lambda f(x'^*) + (1 - \lambda)f(x^*) = f(x^*),$$

which is a contradiction. Therefore,  $x^* = x'^*$ .

**Definition 2.1.1.**

A function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is called *elliptic* if it is of class  $C^1$  and if there exists  $\alpha > 0$  such that

$$\langle \nabla f(x) - \nabla f(y), x - y \rangle \geq \alpha \|x - y\|^2, \quad \forall x, y \in \mathbb{R}^n.$$

**Corollary 2.1.1.**

If  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  is differentiable, then the following are equivalent:

1.  $f$  is  $\alpha$ -elliptic.
2.  $f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle + \frac{\alpha}{2} \|y - x\|^2, \quad \forall x, y \in \mathbb{R}^2.$
3.  $\langle \nabla f(y) - \nabla f(x), y - x \rangle \geq \alpha \|y - x\|^2, \quad \forall x, y \in \mathbb{R}^2.$

**Theorem 2.1.4 (Existence and Uniqueness) [11]**

Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be of class  $C^1$  and  $\alpha$ -elliptic, i.e., there exists  $\alpha > 0$  such that

$$\langle \nabla f(x) - \nabla f(y), x - y \rangle \geq \alpha \|x - y\|^2, \quad \forall x, y \in \mathbb{R}^n.$$

Then  $f$  is strictly convex and coercive. In particular, problem (2.1) admits a unique solution.

**Proof.**

It is clear that ellipticity implies strict convexity, which in turn implies convexity. If  $f$  is  $\alpha$ -elliptic and differentiable, using the previous characterization, we obtain

$$\langle \nabla f(x) - \nabla f(0), x \rangle \geq \alpha \|x\|^2,$$

which implies that  $f$  is coercive.

**Theorem 2.1.5[11]**

Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be a function that is bounded below, lower semi-continuous, and coercive. Then, problem (I) admits at least one solution [11, 13].

**Proof.**

Since  $f$  is bounded below, it cannot take the value  $-\infty$ . Thus, there exists a minimizing sequence  $(x_k)$  of  $f$  such that:

$$\inf_{x \in \mathbb{R}^n} f(x) = \lim_{k \rightarrow +\infty} f(x_k) = d > -\infty.$$

Moreover, since  $f$  is not identically  $+\infty$ , we have  $d < +\infty$ . Let us show that the sequence  $(x_k)$  is bounded. Suppose, for contradiction, that  $(x_k)$  is unbounded. Then

$$\|x_k\| \rightarrow +\infty,$$

and since  $f$  is coercive,  $\lim_{k \rightarrow +\infty} f(x_k) = +\infty$ , which contradicts  $\lim_{k \rightarrow +\infty} f(x_k) = d$ . Therefore,  $(x_k)$  is bounded.

We can extract a convergent subsequence, say  $(x_{k_l})$ , converging to  $x^*$ . Since  $f$  is lower semi-continuous, we have:

$$\liminf_{l \rightarrow +\infty} f(x_{k_l}) \geq f(x^*).$$

But  $\lim_{l \rightarrow +\infty} f(x_{k_l}) = d$ , thus  $f(x^*) \leq d$ . By definition of  $d$ , we also have  $f(x^*) \geq d$ . Therefore,

$$f(x^*) = d = \inf_{x \in \mathbb{R}^n} f(x).$$

**Proposition 2.1.1 (Characterization of ellipticity using the Hessian).**

Let  $f$  be of class  $C^2$ . Then  $f$  is elliptic if and only if there exists  $\beta > 0$  such that

$$\langle \nabla^2 f(x)h, h \rangle \geq \beta \|h\|^2, \quad \forall x, h \in \mathbb{R}^n.$$

**Proof.**

Assume  $f$  is elliptic. Fix  $h \in \mathbb{R}^n$  and define  $g(x) = \langle \nabla f(x), h \rangle$ . Then

$$\langle \nabla^2 f(x)h, h \rangle = \langle \nabla g(x), h \rangle = \frac{\partial g}{\partial h}(x) = \lim_{t \rightarrow 0} \frac{\langle \nabla f(x+th), h \rangle - \langle \nabla f(x), h \rangle}{t}.$$

Using bilinearity and ellipticity:

$$\langle \nabla^2 f(x)h, h \rangle = \lim_{t \rightarrow 0} \frac{\langle \nabla f(x+th) - \nabla f(x), th \rangle}{t^2} \geq \alpha \|h\|^2.$$

Hence (2.2) holds with  $\beta = \alpha$ .

Conversely, suppose (2.2) holds. Let  $x, y \in \mathbb{R}^n$  and define  $g_1(z) = \langle \nabla f(z), x - y \rangle$ . Then

$$\langle \nabla f(x) - \nabla f(y), x - y \rangle = g_1(x) - g_1(y) = \langle \nabla g_1(y + \theta(x - y)), x - y \rangle$$

for some  $\theta \in (0, 1)$ . Moreover,

$$\nabla g_1(z) = \nabla^2 f(z)(x - y),$$

thus,

$$\langle \nabla f(x) - \nabla f(y), x - y \rangle = \langle \nabla^2 f(y + \theta(x - y))(x - y), x - y \rangle \geq \beta \|x - y\|^2.$$

Therefore,  $f$  is elliptic with  $\alpha = \beta$ .

**Examples of elliptic functions.**

1. **Case  $n = 1$ :** Any function  $f : \mathbb{R} \rightarrow \mathbb{R}$  of class  $C^2$  satisfying  $\exists \alpha > 0$  such that  $f''(x) \geq \alpha, \forall x \in \mathbb{R}$  is elliptic.

(a)  $f(x) = ax^2 + bx + c$  with  $a > 0$ .

(b)  $f(x) = x^2 + \sin(x)$  since  $f''(x) = 2 - \sin(x) \geq 1$ .

2. **General case ( $n \geq 1$ ):** Let

$$f(x) = \frac{1}{2} \langle Ax, x \rangle - \langle b, x \rangle + c,$$

with  $A$  symmetric real matrix,  $b \in \mathbb{R}^n, c \in \mathbb{R}$ . Then

$$\nabla f(x) = Ax - b, \quad \nabla^2 f(x) = A.$$

As  $A$  is symmetric, its eigenvalues are real and

$$\langle Ah, h \rangle \geq \lambda_{\min} \|h\|^2,$$

where  $\lambda_{\min}$  is the smallest eigenvalue of  $A$ .

$A$  is positive definite iff  $\lambda_{\min} > 0$ . Thus, by the proposition,  $f$  is elliptic iff  $A$  is positive definite.

**Theorem 2.1.6: Characterization of Ellipticity [13]**

Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be a function that is twice differentiable in the Gateaux sense. Then,  $f$  is **elliptic** if and only if:

$$\exists \alpha > 0, \quad \forall x, h \in \mathbb{R}^n, \quad \langle \nabla^2 f(x)h, h \rangle \geq \alpha \|h\|^2.$$

**Proof.**

( $\Rightarrow$ ) Assume that  $f$  is elliptic. Then by the definition of ellipticity, we have:

$$\langle \nabla f(x) - \nabla f(y), x - y \rangle \geq \alpha \|x - y\|^2, \quad \forall x, y \in \mathbb{R}^n.$$

Taking  $y = x + th$  and defining  $\phi(t) = \nabla f(x + th)$ , we write:

$$\langle \phi(t) - \phi(0), th \rangle \geq \alpha t^2 \|h\|^2.$$

Dividing both sides by  $t^2$  and taking the limit as  $t \rightarrow 0$ , we get:

$$\lim_{t \rightarrow 0} \frac{\langle \phi(t) - \phi(0), th \rangle}{t^2} = \langle \nabla^2 f(x)h, h \rangle \geq \alpha \|h\|^2.$$

( $\Leftarrow$ ) Conversely, assume that:

$$\langle \nabla^2 f(x)h, h \rangle \geq \alpha \|h\|^2, \quad \forall x, h \in \mathbb{R}^n.$$

Let  $x, y \in \mathbb{R}^n$  be arbitrary, and define  $\gamma(t) = x + t(y - x)$  for  $t \in [0, 1]$ . Then,

$$\nabla f(y) - \nabla f(x) = \int_0^1 \nabla^2 f(\gamma(t))(y - x) dt.$$

Hence,

$$\langle \nabla f(y) - \nabla f(x), y - x \rangle = \int_0^1 \langle \nabla^2 f(\gamma(t))(y - x), y - x \rangle dt \geq \int_0^1 \alpha \|y - x\|^2 dt = \alpha \|y - x\|^2.$$

Thus,  $f$  is elliptic.

**Proposition 2.3.1** (Strong monotonicity of the gradient). *If  $f$  is  $\alpha$ -strongly convex on  $K$ , then its gradient is  $\alpha$ -strongly monotone:*

$$(\nabla f(y) - \nabla f(x)) \cdot (y - x) \geq \alpha \|y - x\|^2, \quad \forall x, y \in K.$$

**Proposition 2.3.2** (Uniqueness of the minimizer). *If  $f$  is  $\alpha$ -strongly convex on  $K$ , then  $f$  has at most one minimizer in  $K$ .*

**Remark 2.3.1.** *The three equivalent statements in the theorem provide alternative ways to verify strong convexity: (i) directly from the definition, (ii) via a quadratic lower bound, and (iii) through the strong monotonicity of the gradient. This last condition is especially useful in optimization theory.*

## 2.4 Optimality condition

### 2.4.1 First order optimality condition

In this section, we focus on first-order optimality conditions. The objective is to determine criteria that characterize candidate points for being optimal solutions of a minimization problem.

More precisely, we will establish the necessary and/or sufficient conditions that a function must satisfy at a given point for it to be a local minimum. These conditions are based essentially on the analysis of the function's first derivative (gradient).

We will begin by rigorously defining the notion of a critical point, then we will state and prove the results related to first-order conditions, emphasizing their assumptions and their geometric and analytical interpretation.

### 2.4.2 First-order Necessary Optimality Conditions: NOC1

Given a point  $x^*$ , the continuous differentiability property of the function  $f$  provides a first way to characterize an optimal solution.

**Theorem 2.1.7 [11] [13]**

Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be differentiable at the point  $x^* \in \mathbb{R}^n$ . If  $x^*$  is a local minimum of (2.1), then

$$\nabla f(x^*) = 0.$$

### Proof.

We proceed by contradiction. Suppose that  $\nabla f(x^*) \neq 0$ . Then, the vector  $d = -\nabla f(x^*)$  is a descent direction, i.e.,

$$\nabla f(x^*)^T d = -\|\nabla f(x^*)\|^2 < 0.$$

By the descent direction theorem, there exists  $\delta > 0$  such that

$$f(x^* + \alpha d) < f(x^*), \quad \forall \alpha \in ]0, \delta[.$$

This contradicts the fact that  $x^*$  is a local minimum. Therefore,  $\nabla f(x^*) = 0$ .

### 2.4.3 First-order Necessary and Sufficient Optimality Condition: NSOC1

The convexity of  $f$  allows us to specify the nature (maximum or minimum) of the considered extrema. Of course, similar results can be stated for relative maxima. If  $f$  is convex, the first-order necessary condition is also sufficient.

**Theorem 2.1.8[11]**

Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be a convex function differentiable at the point  $x^* \in \mathbb{R}^n$ . Then  $x^*$  is a minimum of (2.1) if and only if

$$\nabla f(x^*) = 0.$$

**Proof.**

We have already seen that this condition is always necessary. Let us prove that it is sufficient.

Assume that  $x^* \in \mathbb{R}^n$  satisfies  $\nabla f(x^*) = 0$ . Since  $f$  is convex, we have:

$$\forall x \in \mathbb{R}^n, \quad f(x) \geq f(x^*) + \nabla f(x^*)^T(x - x^*) \Rightarrow f(x) \geq f(x^*).$$

Thus,  $x^*$  is a global minimum of  $f$  over  $\mathbb{R}^n$ .

## 2.5 Second-order optimality conditions

The following results are stated for relative minima involving second derivatives.

### 2.5.1 Necessary second-order optimality condition: NOC2

**Theorem 2.1.9.** Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be twice differentiable at  $x^* \in \mathbb{R}^n$ . If  $f$  has a local minimum at  $x^*$ , then:

1.  $\nabla f(x^*) = 0$ , 2. The Hessian matrix  $H(x^*)$  is positive semi-definite.

**Proof.** Using second-order Taylor expansion,

$$f(x^* + \lambda d) = f(x^*) + \lambda \nabla f(x^*)^T d + \frac{1}{2} \lambda^2 d^T H(x^*) d + \lambda^2 \|d\|^2 \alpha(x^*, \lambda d),$$

where  $\alpha(x^*, \lambda d) \rightarrow 0$  as  $\lambda \rightarrow 0$ . Since  $x^*$  is a local minimum,  $\nabla f(x^*) = 0$ , thus

$$\frac{1}{2} d^T H(x^*) d \geq 0, \quad \forall d \in \mathbb{R}^n.$$

Therefore,  $H(x^*)$  is positive semi-definite. □

### 2.5.2 Sufficient second-order optimality condition: COS2

**Theorem 2.1.10[11]** Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be twice differentiable at  $x^* \in \mathbb{R}^n$ . If

1.  $\nabla f(x^*) = 0$ , 2.  $H(x^*)$  is positive definite,
- then  $x^*$  is a local minimum.

**Proof.** Using second-order Taylor expansion,

$$f(x) = f(x^*) + \nabla f(x^*)^T(x - x^*) + \frac{1}{2}(x - x^*)^T H(x^*)(x - x^*) + \|x - x^*\|^2 \alpha(x^*, x - x^*),$$

where  $\alpha(x^*, x - x^*) \rightarrow 0$  as  $x \rightarrow x^*$ . If  $H(x^*)$  is positive definite, the quadratic term is strictly positive. Thus,  $x^*$  is a strict local minimum. □

## Example

Consider  $f(x, y) = (x - 5)^2 + (y - 2)^2$ .

**Step 1.** Find critical points:

$$f'_x = 2(x - 5) = 0 \Rightarrow x = 5, \quad f'_y = 2(y - 2) = 0 \Rightarrow y = 2.$$

**Step 2.** Analyze the Hessian:

$$H = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

Determinant is  $4 > 0$  with positive eigenvalues. Thus  $(5, 2)$  is a local minimum.

**Reference.** See Rockafellar (1970) *Convex Analysis*.

## 2.6 Theorem: Global Optimality of Local Minima for Convex Functions

**Theorem 2.1.11[11]**, Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be a convex function.

1. Every local minimum of  $f$  is also a global minimum.
2. Furthermore, if  $f$  is differentiable, every stationary point  $x^*$  (i.e.,  $\nabla f(x^*) = 0$ ) is a global minimum of  $f$ .

**Proof.**

(i) Let  $x^*$  be a local minimum. For any  $x \in \mathbb{R}^n$ , define  $x_\lambda = \lambda x + (1 - \lambda)x^*$ . By local minimality, for  $\lambda$  small enough,

$$f(x^*) \leq f(x_\lambda).$$

By convexity,

$$f(x_\lambda) \leq \lambda f(x) + (1 - \lambda)f(x^*).$$

Thus,

$$f(x^*) \leq \lambda f(x) + (1 - \lambda)f(x^*),$$

which implies

$$0 \leq \lambda(f(x) - f(x^*)),$$

and hence  $f(x^*) \leq f(x)$  for all  $x$ .

(ii) If  $f$  is differentiable and  $\nabla f(x^*) = 0$ , by the first-order condition of convex functions,

$$f(x) \geq f(x^*) + \nabla f(x^*)^T(x - x^*) = f(x^*),$$

so  $x^*$  is a global minimum. □

**Reference.** See Rockafellar (1970), Theorem 27.1; Boyd and Vandenberghe (2004), Section 3.1.2.

## Exercise

**Exercise 2.1.6.** Let the function  $f$  be defined by:

$$f(x, y) = (y - 1) \ln(y - 1) - \ln(x) + x^2 - xy + 2y^2 - 7y - \frac{3}{2}x + 3.$$

- (a) Give the domain of definition  $D_f$  of  $f$  and draw this set.
- (b) Is the set  $D_f$  convex? Is it open?
- (c) Show that the function  $\varphi : u \rightarrow \ln(u)$  is convex on its domain.
- (d) Deduce the convexity of  $f$  on  $D_f$ .
- (e) Show that  $(2, 2)$  is a critical point.
- (f) Deduce the nature of  $(2, 2)$ .
- (g) Does the function  $f$  admit a global maximum on  $D_f$ ?

## Solution

Define:

$$f(x, y) = (y - 1) \ln(y - 1) - \ln(x) + x^2 - xy + 2y^2 - 7y - \frac{3}{2}x + 3.$$

- (a) **Domain:**

$x > 0$  and  $y > 1$ , thus:

$$D_f = \{(x, y) \in \mathbb{R}^2 \mid x > 0, y > 1\}.$$

- (b) **Convexity and openness:**

Domain is convex (intersection of half-planes) and open.

- (c) **Convexity of  $\varphi(u) = \ln(u)$ :**

$\varphi''(u) = -\frac{1}{u^2} < 0$ , thus concave.

- (d) **Convexity of  $f$ :**

Due to quadratic terms (full proof via Hessian positivity).

- (e) **Critical point at  $(2, 2)$ :**

$$\frac{\partial f}{\partial x} = -\frac{1}{2} + 4 - 2 - \frac{3}{2} = 0,$$

$$\frac{\partial f}{\partial y} = \ln(1) + 1 - 2 - 7 + 8 = 0.$$

- (f) **Nature:**

Positive definite Hessian implies strict local minimum (also global by convexity).

- (g) **Global maximum:**

No global maximum, function tends to  $+\infty$  on  $D_f$ .

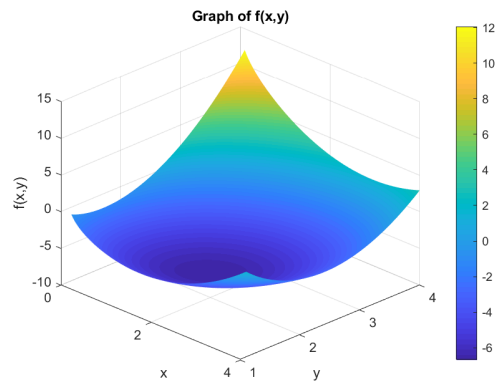


Figure 2.1: Graph of  $f(x, y)$