Sheet 01

Exercise 1

Let the following functions be:

1. $f_1(x) = \dfrac{3}{2}x_1^2 + 2x_2^2 + \dfrac{3}{2}x_3^2 + x_1x_2 + 2x_2x_3 - 3x_1 - x_3$
2. $f_2(x) = (x_1 - 1)^2 + 10(x_1 - x_2)^2$
3. $f_3(x) = 5x_1^2 + 5x_2^2 - x_1x_2 + 11x_1 + 11x_2 + 11$

(a) Calculate $\nabla f_i(x)$ and $\nabla^2 f_i(x)$ for $i = 1, 2, 3$.

(b) Among these functions, which are quadratic? Justify.

—

Exercise 2

Determine $Df(x)$ and $\nabla^2 f(x)$ of the quadratic function:

$$f(x) = \frac{1}{2}\langle x, Ax\rangle + \langle x, b\rangle + c,$$

where $A \in M_{n\times n}(\mathbb{R})$, $b \in \mathbb{R}^n$, and $c \in \mathbb{R}$.

—

Exercise 3

Give the second-order Taylor expansion of function f around point x_0 for:

(a) $f(x) = x_1e^{-x_2} + x_2 + 1$, $x_0 = (1, 0)^T$.

(b) $f(x) = x_1^4 - 2x_1^2x_2^2 + x_2^4$, $x_0 = (1, 1)^T$.

—

Exercise 4

Let $x(t) = (t^2, t, t)^T$, $t \in \mathbb{R}$, and $f(x) = x_1x_2^3 + x_1x_2 + x_3$, with $x = (x_1, x_2, x_3)^T \in \mathbb{R}^3$. Find $\frac{d}{dt}f(x(t))$ in terms of t.

—

Exercise 5

The purpose of this exercise is to derive Taylor's formula for a function $f : \mathbb{R}^n \to \mathbb{R}$. Let $f \in C^2$, $x, x_0 \in \mathbb{R}^n$, and define:

$$z(\alpha) = x_0 + \alpha\frac{(x - x_0)}{\|x - x_0\|}.$$

Define the function $\Phi(\alpha) = f(z(\alpha))$.

(a) Determine $\Phi'(\alpha)$ and $\Phi''(\alpha)$.

(b) By noting that $f(x) = \Phi(\|x - x_0\|)$, deduce:

$$f(x) = f(x_0) + Df(x_0)(x - x_0) + \frac{1}{2}(x - x_0)^T \nabla^2 f(x_0)(x - x_0) + o(\|x - x_0\|^2).$$

—

Exercise 6

Let ϕ be a continuous function from $\mathbb{R}$ to $\mathbb{R}$, and define $f, g : \mathbb{R}^2 \to \mathbb{R}$ by:

$$f(x,y) = \int_0^{x+y} \phi(t)\,dt, \quad g(x,y) = \int_0^{xy} \phi(t)\,dt.$$

(a) Show that f and g are of class C^1 on $\mathbb{R}^2$.

(b) Let $Df(x,y)$ and $Dg(x,y)$ be the differentials of f and g at point (x,y), respectively. Calculate $Df(x,y)(h,k)$ and $Dg(x,y)(h,k)$ for all $(h,k) \in \mathbb{R}^2$.

—

Exercise 7

Let $f : \mathbb{R}^n \to \mathbb{R}$ and $x : \mathbb{R} \to \mathbb{R}^n$ be two C^2 functions. Define $g(t) = f(x(t))$.

(a) Calculate $g''(t)$ in the case where $x(t) = u + tv$, where $u, v \in \mathbb{R}^n$.

(b) Calculate $g''(t)$ for general $x(t)$.

—

Exercise 8

Which of the following sets are convex?

1. $S_1 = \{(x,y) \in \mathbb{R}^2 \mid 0 \le x \le 1, y = 0\}$
2. $S_2 = \{(x,y) \in \mathbb{R}^2 \mid x \ge 0, y \ge 0, x + y \le 1\}$
3. $S_3 = \{x \in \mathbb{R}^n \mid A_1 x = b_1, A_2 x \le b_2\}$ where A_1 and A_2 are matrices of size $m \times n$, and b_1 and b_2 are vectors in $\mathbb{R}^m$.
4. $S_4 = \{(x,y) \in \mathbb{R}^2 \mid y - x^2 \ge 0\}$
5. $S_5 = \{(x,y) \in \mathbb{R}^2 \mid xy \ge 1, x > 0\}$

—

Exercise 9

Verify whether the following functions are convex or not on $\mathbb{R}^2$:

1. $f(x,y) = x^2 - xy + 2y^2 - 2x + e^{x+y}$
2. $f(x,y) = (x-2)^4 + (x-2)^2 y^2 + (y+1)^2$
3. $f(x,y) = -x^2 - 2xy - 2y^2$

—

Exercise 10

We consider the function f defined on $\mathbb{R}^2$ by

$$f(x,y) = x^4 + y^4 - 2(x-y)^2.$$

1. Show that there exist $(a,b) \in \mathbb{R}_+^2$ (and determine them) such that

   $$f(x,y) \geq a\|(x,y)\|^2 + b$$

   for all $(x,y) \in \mathbb{R}^2$, where $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^2$. Deduce that the problem

   $$\inf_{(x,y)\in\mathbb{R}^2} f(x,y)$$

   has at least one solution.

2. Is the function f convex on $\mathbb{R}^2$?

—

Exercise 11

We define the function $J : \mathbb{R}^2 \to \mathbb{R}$ by

$$J(x,y) = y^4 - 3xy^2 + x^2.$$

1. Determine the critical points of J.

2. Let $d = (d_1, d_2) \in \mathbb{R}^2$. Using the function $t \mapsto J(td_1, td_2)$, show that $(0,0)$ is a local minimum along any line passing through $(0,0)$.

3. Is the point $(0,0)$ a local minimum of the restriction of J to the parabola given by the equation $x = y^2$?

4. Compute the Hessian matrix of J. What is the nature of the critical point $(0,0)$?

Solution (Sheet 01)

Solution 1.1

Let us consider the following functions:

1.

$$f_1(x) = \frac{3}{2}x_1^2 + 2x_2^2 + \frac{3}{2}x_3^2 + x_1x_2 + 2x_2x_3 - 3x_1 - x_3.$$

2.

$$f_2(x) = (x_1 - 1)^2 + 10(x_1 - x_2)^2.$$

3.

$$f_3(x) = 5x_1^2 + 5x_2^2 - x_1x_2 + 11x_1 + 11x_2 + 11.$$

(a) Compute $\nabla f_i(x)$ and $\nabla^2 f_i(x)$ for $i = 1, 2, 3$.

- For $f_1(x)$:

$$\nabla f_1(x) = \begin{pmatrix} 3x_1 + x_2 - 3 \\ 4x_2 + x_1 + 2x_3 \\ 3x_3 + 2x_2 - 1 \end{pmatrix}, \quad \nabla^2 f_1(x) = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 4 & 2 \\ 0 & 2 & 3 \end{pmatrix}.$$

- For $f_2(x)$:

$$\nabla f_2(x) = \begin{pmatrix} 22x_1 - 20x_2 - 2 \\ -20x_1 + 20x_2 \end{pmatrix}, \quad \nabla^2 f_2(x) = \begin{pmatrix} 22 & -20 \\ -20 & 20 \end{pmatrix}.$$

- For $f_3(x)$:

$$\nabla f_3(x) = \begin{pmatrix} 10x_1 - x_2 + 11 \\ -x_1 + 10x_2 + 11 \end{pmatrix}, \quad \nabla^2 f_3(x) = \begin{pmatrix} 10 & -1 \\ -1 & 10 \end{pmatrix}.$$

(b) Since $\nabla^2 f_i(x)$ is constant and symmetric, each f_i is quadratic for $i = 1, 2, 3$.

—

Solution 1.2

Clearly, the function f is of class C^2, so we can apply Lemma 1.1:

$$Df(x)h = \lim_{t \to 0} \frac{f(x + th) - f(x)}{t}.$$

We have:

$$f(x + th) = \frac{1}{2}\langle x + th, A(x + th)\rangle + \langle b, x + th\rangle + c.$$

Expanding,

$$= f(x) + t\langle Ax, h\rangle + t\langle b, h\rangle + \frac{t^2}{2}\langle Ah, h\rangle.$$

Hence,

$$Df(x)h = \langle Ax, h\rangle + \langle b, h\rangle = \langle Ax + b, h\rangle.$$

Therefore,

$$Df(x) = Ax + b.$$

—

Solution 1.3

Given $x(t) = (e^t, t^2, t)^T$ and $f(x) = x_1x_2x_3 + x_1x_2 + x_3$, compute $\frac{d}{dt}f(x(t))$.

Theorem. If $f : D \subset \mathbb{R}^n \to \mathbb{R}$ is differentiable, and $x : (a,b) \to D$ is differentiable, then

$$\frac{d}{dt}f(x(t)) = \nabla f(x(t))^T x'(t).$$

Using this,

$$\nabla f(x) = \begin{pmatrix} x_2x_3 + x_2 \\ x_1x_3 + x_1 \\ x_1x_2 + 1 \end{pmatrix}.$$

Thus,

$$x'(t) = \begin{pmatrix} e^t \\ 2t \\ 1 \end{pmatrix}.$$

Therefore,

$$\frac{d}{dt}f(x(t)) = (x_2x_3 + x_2)e^t + (x_1x_3 + x_1)2t + (x_1x_2 + 1).$$

—

Solution 1.4

Recall (Taylor expansion of order 2):

If $f \in C^2$, then near x_0,

$$f(x) = f(x_0) + Df(x_0)(x - x_0) + \frac{1}{2}(x - x_0)^T \nabla^2 f(x_0)(x - x_0) + o(\|x - x_0\|^2).$$

(a)

For $f(x) = x_1 e^{-x_2} + x_2 + 1$, at $x_0 = (1,0)^T$,

Calculate gradients and Hessian and write the expansion accordingly (details skipped here).

—

Solution 1.5

Given $z(\alpha) = x_0 + \alpha\frac{x - x_0}{\|x - x_0\|}$ and $\Phi(\alpha) = f(z(\alpha))$.

(a) Using chain rule,

$$\Phi'(\alpha) = \frac{1}{\|x - x_0\|}(x - x_0)^T \nabla f(z(\alpha)),$$

$$\Phi''(\alpha) = \frac{1}{\|x - x_0\|^2}(x - x_0)^T \nabla^2 f(z(\alpha))(x - x_0).$$

(b) Hence,

$$f(x) = f(x_0) + Df(x_0)(x - x_0) + \frac{1}{2}(x - x_0)^T \nabla^2 f(x_0)(x - x_0) + o(\|x - x_0\|^2).$$

—

Solution 1.6

Define $\phi : \mathbb{R} \to \mathbb{R}$ continuous, and

$$f(x,y) = \int_0^{x+y} \phi(t)dt, \quad g(x,y) = \int_0^{xy} \phi(t)dt.$$

1. Both f and g are C^1 as compositions of C^1 functions.
2. Their differentials are:

$$Df(x,y)(h,k) = \phi(x+y)(h+k),$$

$$Dg(x,y)(h,k) = \phi(xy)(yh+xk).$$

—

Exercise 1.7

1. Let $S_1 = \{(x,y) \in \mathbb{R}^2 \mid y - x \geq 0\}$. To prove S_1 is convex, let $X_1 = (x_1, y_1), X_2 = (x_2, y_2) \in S_1$ and $\lambda \in [0,1]$. Then:

$$\lambda X_1 + (1-\lambda)X_2 = (\lambda x_1 + (1-\lambda)x_2, \lambda y_1 + (1-\lambda)y_2).$$

We need:

$$\lambda y_1 + (1-\lambda)y_2 - [\lambda x_1 + (1-\lambda)x_2] \geq 0,$$

which simplifies to:

$$\lambda(y_1 - x_1) + (1-\lambda)(y_2 - x_2) \geq 0.$$

Since $y_i - x_i \geq 0$, S_1 is convex.

—

2. Let $S_2 = \{(x,y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0, x + y \leq 1\}$. For any $X_1, X_2 \in S_2$, $\lambda \in [0,1]$, define:

$$X = \lambda X_1 + (1-\lambda)X_2.$$

We check:

$$x \geq 0, \quad y \geq 0, \quad x + y \leq 1.$$

Indeed,

$$x + y = \lambda(x_1 + y_1) + (1-\lambda)(x_2 + y_2) \leq \lambda \cdot 1 + (1-\lambda) \cdot 1 = 1.$$

Hence S_2 is convex.

—

3. Let $S_3 = \{x \in \mathbb{R}^n \mid A_1 x = b_1, A_2 x \leq b_2\}$. For any $x, y \in S_3$ and $\lambda \in [0,1]$:

$$A_1(\lambda x + (1-\lambda)y) = \lambda A_1 x + (1-\lambda)A_1 y = b_1,$$

$$A_2(\lambda x + (1-\lambda)y) \leq \lambda A_2 x + (1-\lambda)A_2 y \leq b_2.$$

Thus, S_3 is convex.

—

4. Let $S_4 = \{(x,y) \in \mathbb{R}^2 \mid y - x^2 \geq 0\}$. For any $X, Y \in S_4$ and $\lambda \in [0,1]$:

$$\lambda y_1 + (1-\lambda)y_2 \geq \lambda x_1^2 + (1-\lambda)x_2^2.$$

By Jensen's inequality:

$$(\lambda x_1 + (1-\lambda)x_2)^2 \leq \lambda x_1^2 + (1-\lambda)x_2^2.$$

Hence:

$$\lambda y_1 + (1-\lambda)y_2 \geq (\lambda x_1 + (1-\lambda)x_2)^2,$$

so S_4 is convex.

—

5. Let $S_5 = \{(x,y) \in \mathbb{R}^2 \mid xy \geq 1, x > 0\}$. For $X, Y \in S_5$ and $\lambda \in [0,1]$:

$$(\lambda x_1 + (1-\lambda)x_2)(\lambda y_1 + (1-\lambda)y_2) \geq \lambda^2 x_1 y_1 + (1-\lambda)^2 x_2 y_2 + \lambda(1-\lambda)(x_1 y_2 + x_2 y_1).$$

However, this is not necessarily ≥ 1, so S_5 is not convex.

—

Exercise 1.8

1. For $f(x,y) = x^2 - xy + 2y^2 - 2x + e^{x+y}$:

$$\nabla f(x,y) = \begin{pmatrix} 2x - y - 2 + e^{x+y} \\ -x + 4y + e^{x+y} \end{pmatrix},$$

$$\nabla^2 f(x,y) = \begin{pmatrix} 2 + e^{x+y} & -1 + e^{x+y} \\ -1 + e^{x+y} & 4 + e^{x+y} \end{pmatrix}.$$

The eigenvalues are positive, so f is strictly convex.

—

2. For $f(x,y) = (x-2)^4 + (x-2)^2 y^2 + (y+1)^2$:
The Hessian is:

$$\nabla^2 f(x,y) = \begin{pmatrix} 12(x-2)^2 + 2y^2 & 4y(x-2) \\ 4y(x-2) & 2(x-2)^2 + 2 \end{pmatrix}.$$

The sign changes depending on x, y, so f is neither convex nor concave.

—

3. For $f(x,y) = -x^2 - 2xy - 2y^2$:
The Hessian is:

$$\nabla^2 f(x,y) = \begin{pmatrix} -2 & -2 \\ -2 & -4 \end{pmatrix}.$$

The eigenvalues are negative, so f is strictly concave.

—

Exercise 1.9

1. The function f is polynomial, thus of class $\mathcal{C}^\infty(\mathbb{R}^2)$. It is given by:

$$f(x,y) = x^4 + y^4 - 2(x-y)^2.$$

To prove that f is coercive, note that:

$$x^4 + y^4 \geq \frac{1}{2}(x^2 + y^2)^2 \geq 0.$$

Also,

$$-2(x-y)^2 \geq -4(x^2 + y^2).$$

Thus,

$$f(x,y) \geq \frac{1}{2}(x^2+y^2)^2 - 4(x^2+y^2).$$

Define $r^2 = x^2 + y^2$, then

$$f(x,y) \geq \frac{1}{2}r^4 - 4r^2.$$

As $r \to \infty$, $f(x,y) \to \infty$. Therefore, f is coercive and attains a minimum on $\mathbb{R}^2$.

—

2. To check convexity, compute the Hessian:

$$\nabla^2 f(x,y) = \begin{pmatrix} 12x^2 - 4 & 0 \\ 0 & 12y^2 - 4 \end{pmatrix}.$$

At $(0,0)$, the eigenvalues are -4 and -4, so the Hessian is not positive semidefinite everywhere. Thus, f is not convex.

—

Exercise 1.10

Let $J(x,y) = y^4 - 3xy^2 + x^2$.

1. Compute $\nabla J(x,y)$:

$$\nabla J(x,y) = \begin{pmatrix} -3y^2 + 2x \\ 4y^3 - 6xy \end{pmatrix}.$$

Set $\nabla J = 0$, then:

$$-3y^2 + 2x = 0 \quad \Rightarrow \quad x = \frac{3}{2}y^2,$$

$$4y^3 - 6xy = 0.$$

Substitute x:

$$4y^3 - 6\left(\frac{3}{2}y^2\right)y = 4y^3 - 9y^3 = -5y^3 = 0.$$

Thus, $y = 0$, so $x = 0$. Therefore, the only critical point is $(0,0)$.

—

2. Consider $t \mapsto J(td_1, td_2)$. Then:

$$J(td_1, td_2) = (td_2)^4 - 3(td_1)(td_2)^2 + (td_1)^2.$$

At $t = 0$, $J(0,0) = 0$. The second derivative is positive along any line through the origin, so $(0,0)$ is a local minimum along any such line.

—

3. Restricting to the parabola $x = y^2$:

$$J(y^2, y) = y^4 - 3(y^2)y^2 + (y^2)^2 = y^4 - 3y^4 + y^4 = -y^4.$$

Thus, $J(y^2, y) = -y^4 \leq 0$ with equality only at $y = 0$. Therefore, $(0,0)$ is a local maximum along this curve, not a minimum.

—

4. Compute the Hessian:

$$\nabla^2 J(x,y) = \begin{pmatrix} 2 & -6y \\ -6y & 12y^2 - 6x \end{pmatrix}.$$

At $(0,0)$:

$$\nabla^2 J(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}.$$

This matrix is positive semidefinite with rank 1, thus $(0,0)$ is a saddle point along some directions and minimum along others.

—

Exercise 1.11

We are asked to derive the second-order Taylor expansion of the function f around x_0.

—

(a) Let $f(x) = x_1 e^{-x_2} + x_2 + 1$, with $x_0 = (1,0)^T$.

First, compute the gradient:

$$\nabla f(x) = \begin{pmatrix} e^{-x_2} \\ -x_1 e^{-x_2} + 1 \end{pmatrix}.$$

At x_0:

$$\nabla f(x_0) = \begin{pmatrix} 1 \\ -1+1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Compute the Hessian:

$$\nabla^2 f(x) = \begin{pmatrix} 0 & -e^{-x_2} \\ -e^{-x_2} & x_1 e^{-x_2} \end{pmatrix}.$$

At x_0:

$$\nabla^2 f(x_0) = \begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix}.$$

Thus, the second-order Taylor expansion is:

$$f(x) \approx f(x_0) + \nabla f(x_0)^T (x - x_0) + \frac{1}{2}(x - x_0)^T \nabla^2 f(x_0)(x - x_0).$$

—

(b) Let $f(x) = x_1^4 - 2x_1^2 x_2^2 + x_2^4$, with $x_0 = (1,1)^T$.

Compute the gradient:

$$\nabla f(x) = \begin{pmatrix} 4x_1^3 - 4x_1 x_2^2 \\ -4x_1^2 x_2 + 4x_2^3 \end{pmatrix}.$$

At x_0:

$$\nabla f(x_0) = \begin{pmatrix} 4-4 \\ -4+4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Compute the Hessian:

$$\nabla^2 f(x) = \begin{pmatrix} 12x_1^2 - 4x_2^2 & -8x_1x_2 \\ -8x_1x_2 & -4x_1^2 + 12x_2^2 \end{pmatrix}.$$

At x_0:

$$\nabla^2 f(x_0) = \begin{pmatrix} 12-4 & -8 \\ -8 & -4+12 \end{pmatrix} = \begin{pmatrix} 8 & -8 \\ -8 & 8 \end{pmatrix}.$$

Therefore, the second-order Taylor expansion is:

$$f(x) \approx f(x_0) + \frac{1}{2}(x - x_0)^T \nabla^2 f(x_0)(x - x_0).$$

—

Exercise 1.12
Let $x(t) = (e^t, 2t, 1)^T$, and $f(x) = x_1x_2x_3 + x_1x_2 + x_3$.
We need to compute:

$$\frac{d}{dt} f(x(t)).$$

First compute $\nabla f(x)$:

$$\nabla f(x) = \begin{pmatrix} x_2x_3 + x_2 \\ x_1x_3 + x_1 \\ x_1x_2 + 1 \end{pmatrix}.$$

Then compute $\frac{dx}{dt}$:

$$x'(t) = \begin{pmatrix} e^t \\ 2 \\ 0 \end{pmatrix}.$$

Thus,

$$\frac{d}{dt} f(x(t)) = \nabla f(x(t))^T x'(t).$$

Calculate at general t:

$$= [x_2x_3 + x_2, x_1x_3 + x_1, x_1x_2 + 1] \begin{pmatrix} e^t \\ 2 \\ 0 \end{pmatrix}.$$

Substitute $x_1 = e^t$, $x_2 = 2t$, $x_3 = 1$:
- $x_2x_3 + x_2 = 2t \cdot 1 + 2t = 4t$ - $x_1x_3 + x_1 = e^t \cdot 1 + e^t = 2e^t$ - $x_1x_2 + 1 = e^t \cdot 2t + 1 = 2te^t + 1$.
Thus,

$$\frac{d}{dt} f(x(t)) = 4te^t + 2e^t \cdot 2 + 0 = 4te^t + 4e^t.$$

—

Exercise 1.13
Let $f \in C^2$, $x, x_0 \in \mathbb{R}^n$, and define:

$$z(\alpha) = x_0 + \alpha \frac{(x - x_0)}{\|x - x_0\|}.$$

Define $\Phi(\alpha) = f(z(\alpha))$.
—
1. Compute $\Phi'(\alpha)$ and $\Phi''(\alpha)$.
We have:

$$\Phi'(\alpha) = Df(z(\alpha)) \cdot z'(\alpha).$$

But

$$z'(\alpha) = \frac{(x - x_0)}{\|x - x_0\|}.$$

Hence,

$$\Phi'(\alpha) = \nabla f(z(\alpha))^T \frac{(x - x_0)}{\|x - x_0\|}.$$

—
For $\Phi''(\alpha)$:

$$\Phi''(\alpha) = \frac{d}{d\alpha} \left[\nabla f(z(\alpha))^T \frac{(x - x_0)}{\|x - x_0\|} \right].$$

Since $\frac{(x-x_0)}{\|x-x_0\|}$ is constant,

$$\Phi''(\alpha) = \frac{(x - x_0)^T}{\|x - x_0\|^2} \nabla^2 f(z(\alpha))(x - x_0).$$

—
2. Since $f(x) = \Phi(\|x - x_0\|)$, we can write the Taylor expansion:

$$f(x) = \Phi(0) + \|x - x_0\|\Phi'(0) + \frac{1}{2}\|x - x_0\|^2\Phi''(0) + o(\|x - x_0\|^2).$$

Which implies:

$$f(x) = f(x_0) + Df(x_0)(x - x_0) + \frac{1}{2}(x - x_0)^T \nabla^2 f(x_0)(x - x_0) + o(\|x - x_0\|^2).$$

—
This result rederives the second-order Taylor formula.
Exercise 1.14
Let $\phi : \mathbb{R} \to \mathbb{R}$ be continuous, and define:

$$f(x, y) = \int_0^{x+y} \phi(t)\, dt, \quad g(x, y) = \int_0^{xy} \phi(t)\, dt.$$

—
1. Show that f and g are of class C^1.
Since ϕ is continuous and the integration limit is differentiable, f and g are C^1 by differentiation under the integral sign.

—
2. Compute the differentials:
- For f:

$$Df(x,y)(h,k) = \phi(x+y)(h+k).$$

- For g:

$$Df(x,y)(h,k) = \phi(xy)(yh+xk).$$

—

Therefore:

$$Df(x,y)(h,k) = \phi(x+y)(h+k), \quad Dg(x,y)(h,k) = \phi(xy)(yh+xk).$$

Exercise 9

Let $f : \mathbb{R}^n \to \mathbb{R}$, and $x : \mathbb{R} \to \mathbb{R}^n$ be C^2 functions. Define:

$$g(t) = f(x(t)).$$

—

1. If $x(t) = u + tv$, where $u, v \in \mathbb{R}^n$, then:

$$g'(t) = \nabla f(x(t))^T v,$$

$$g''(t) = v^T \nabla^2 f(x(t)) v.$$

—

2. For general $x(t)$,

$$g'(t) = \nabla f(x(t))^T x'(t),$$

$$g''(t) = x'(t)^T \nabla^2 f(x(t)) x'(t) + \nabla f(x(t))^T x''(t).$$

—

These formulas result from applying the chain rule and product rule for derivatives.