

General Introduction to Unconstrained Optimization

Optimization is a fundamental pillar of applied mathematics, engineering, and the sciences, concerned with finding the best possible solution to a given problem according to specific criteria. Among the many categories of optimization, unconstrained optimization occupies a central position, as it deals with the process of minimizing or maximizing an objective function without any explicit restrictions on the decision variables. In other words, it focuses on optimizing functions defined over an entire space without any constraints limiting feasible solutions. As emphasized by Nocedal and Wright [2], unconstrained optimization serves as the foundational basis for the entire field of optimization, since most constrained optimization methods require solving a sequence of unconstrained subproblems.

The primary goal in unconstrained optimization is to determine the point at which the objective function achieves its minimum or maximum value. In practical terms, this involves adjusting the input variables so that the function reaches its optimal value, a concept that underlies numerous practical applications. For instance, in engineering design, minimizing the weight of a structural component while maintaining sufficient strength often starts with an unconstrained optimization formulation before additional real-world constraints are imposed [3]. Similarly, in economics and finance, maximizing utility or profit functions in idealized models often begins with unconstrained optimization before incorporating market restrictions [18].

Historically, unconstrained optimization problems have been studied extensively due to their mathematical elegance and tractability. Classical approaches, such as gradient based methods, Newton's method, and quasi-Newton algorithms, emerged as efficient solutions for smooth and differentiable functions. Fletcher [5] notes that these methods exploit first and second derivative information to iteratively approach optimal solutions with high precision, making them suitable for a wide range of scientific and engineering computations. Moreover, Nelder and Mead [9] developed direct search methods, like the simplex algorithm, which do not require derivative information and are widely used for optimizing non-differentiable or noisy functions.

The importance of unconstrained optimization has further increased with the rise of data science and machine learning. Training modern neural networks relies on minimizing loss functions, which are typically formulated as large-scale unconstrained optimization problems [6]. Techniques such as Stochastic Gradient Descent (SGD) and adaptive methods like Adam [7] were developed to handle optimization problems with millions of parameters and vast datasets, demonstrating the relevance of unconstrained optimization in contemporary computational fields. Beyond its practical applications, unconstrained optimization also provides fundamental theoretical insights.

Concepts such as gradient, Hessian, and convexity originated and were rigorously studied within the framework of unconstrained problems before being generalized to handle constraints and non-convexities in more complex scenarios. As stated by Bertsekas [4], understanding unconstrained optimization is essential for anyone aiming to grasp the general principles and algorithms of nonlinear programming. Recent research continues to improve the scalability, robustness, and convergence of algorithms in unconstrained optimization. Methods such as the limited-memory BFGS [8] have been particularly impactful for large-scale problems where traditional quasi-Newton methods become computationally infeasible. Furthermore,

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optimization software libraries widely used in academia and industry, including MATLAB, SciPy, and TensorFlow, integrate efficient unconstrained optimization algorithms as core modules to solve diverse real-world problems efficiently. In conclusion, unconstrained optimization remains a fundamental area of optimization theory and practice. Its algorithms, models, and insights are embedded in modern computational tools and methods across disciplines. Whether for solving idealized mathematical problems, performing preliminary engineering designs, or training advanced machine learning models, the role of unconstrained optimization is indispensable.

As Nocedal and Wright [2] emphasize, the continued advancement of optimization algorithms, especially for unconstrained problems, drives progress across the entire spectrum of numerical optimization and its vast applications.

Finally, in this polycopé, we will present three major chapters to build a comprehensive understanding of unconstrained optimization. The first chapter is devoted to essential reminders of differential calculus and convexity, providing the theoretical tools required for optimization analysis. The second chapter focuses on unconstrained minimization, covering fundamental concepts and formulations. The third chapter introduces the main methods and algorithms used to solve unconstrained optimization problems efficiently. We conclude the polycopé with a collection of exercises and practical works (TP) to consolidate the theoretical knowledge and enhance problem-solving skills in real-world contexts.