Chapter III: Work and Energy

III.1. Introduction :

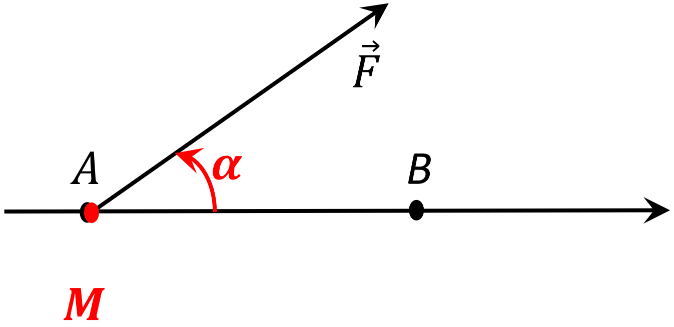
If we know the positions and velocity of the particles of a system and all the forces acting on these particles, we can predict, using Newton's laws, the evolution of this system over time. But in practice, we can't always know all the forces that come into play, and even if we do, the equations to solve are too many or too complicated. For this reason, we appeal to new notions such as “work and energy».

III.2. Work of a Force:

III.2.1. Constant Force on a Straight Displacement:

- A force is said to be constant when his magnitude and direction do not change over time.
- A force is said to work when its point of application moves.

➢ If an object $\boldsymbol{M}$ moves through a rectilinear displacement $\overrightarrow{\boldsymbol{AB}}$ while a constant force $\vec{F}$ is acting on it:

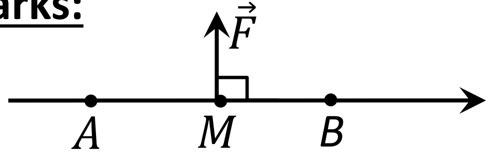


The force does an amount of work equal to:

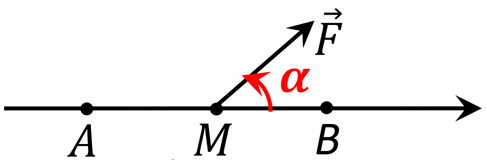
$$W_{\vec{F}} = \vec{F}.\overrightarrow{AB} = F.ABcos\alpha \quad ([W] = Joule)$$

Remarks:

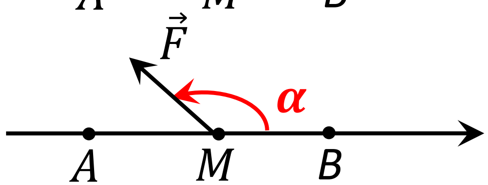
1.



$\alpha = \frac{\pi}{2} \Rightarrow W_{\vec{F}} = 0$: $\vec{F}$ doesn't work



$0 < \alpha < \frac{\pi}{2} \Rightarrow W_{\vec{F}} > 0$: Motor work



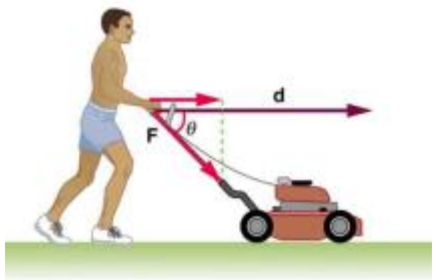
$\frac{\pi}{2} < \alpha < \pi \Rightarrow W_{\vec{F}} < 0$: Resistance work

2. If several different (constant) forces act on a mass while it moves though a displacement $\overrightarrow{\boldsymbol{AB}}$, then we can talk about the **net work** done by the forces:

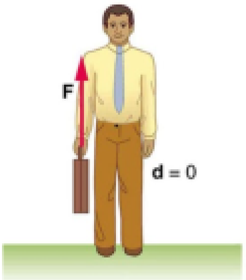
$$W_{net} = \vec{F}_1\overrightarrow{AB} + \vec{F}_2\overrightarrow{AB} + \vec{F}_3\overrightarrow{AB}......+\vec{F}_n\overrightarrow{AB} = \sum_{i=1}^{n} \vec{F}_i.\overrightarrow{AB}$$

Examples of works:

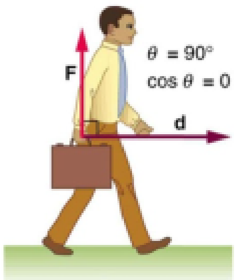
1. The work done bay the force $\vec{F}$ on this lawnmower is $(\boldsymbol{F} \times \boldsymbol{d} \times \boldsymbol{\cos\theta})$



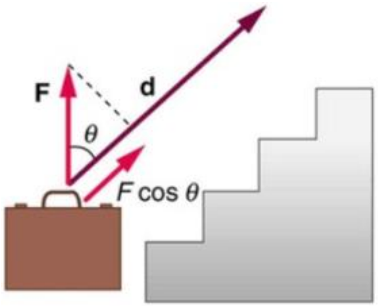
2. A person holding a briefcase does no work on it because there is no motion (d=0)



3. The person moving the briefcase horizontaly at a constant speed deos no work on it.



4. Work is done on the briefcase by carrying it upstairs at a constant speed becasue there is necessarily a component of force F in the direction of the motion.



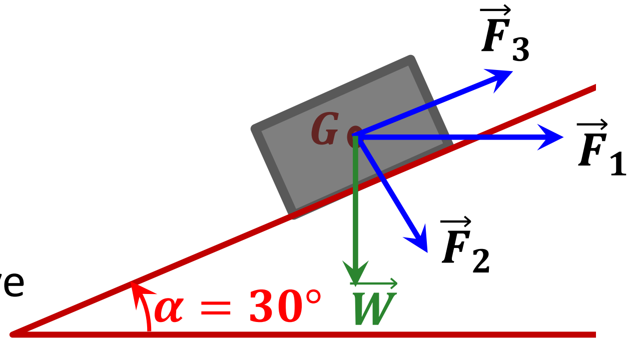
The work done by a constant force can be calculated as the area under the force-displacement graph

Exercise:

A block of stone moves upwards on a plane inclined at 30° under the action of several forces including: $\boldsymbol{F_1 = 45\ N}$ horizontal.

$\boldsymbol{F_2 = 25\ N}$ Normal to the inclined plane.

$\boldsymbol{F_3 = 35\ N}$ parallel to the inclined plane.



It will be considered that all the forces acting on the block have their point of application at the center of mass $\boldsymbol{G}$ of the block.

Calculate the work of forces $\boldsymbol{F_1}$, $\boldsymbol{F_2}$ and $\boldsymbol{F_3}$ when the block rises 1.5 m on the inclined plane.

Solution:

1- $W_1 = \vec{F}_1.\overrightarrow{AB} = F_1.AB.cos\alpha = 45.1{,}5.\cos 30 = 58{,}46J$

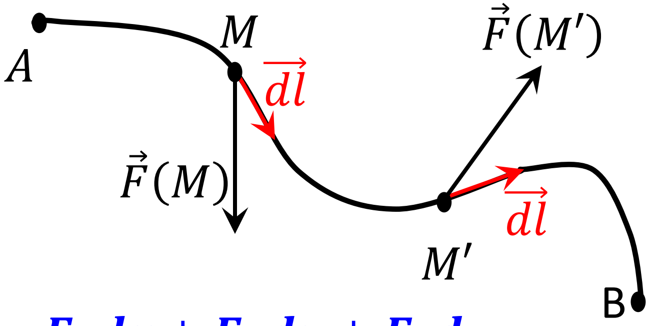
2- $W_2 = \vec{F}_2.\overrightarrow{AB} \quad (\vec{F}_2 \perp \overrightarrow{AB}) \Rightarrow W_2 = 0$

3- $W_3 = \vec{F}_3.\overrightarrow{AB}\ (\vec{F}_3 \parallel \overrightarrow{AB}) \Rightarrow W_3 = F_3.AB = 35.1{,}5 = 52{,}5J$

III.2.2. Elementary work:

- When the force $\vec{F}$ which acts on ***M*** is not constant during displacement:

By definition, elementary work is given by:



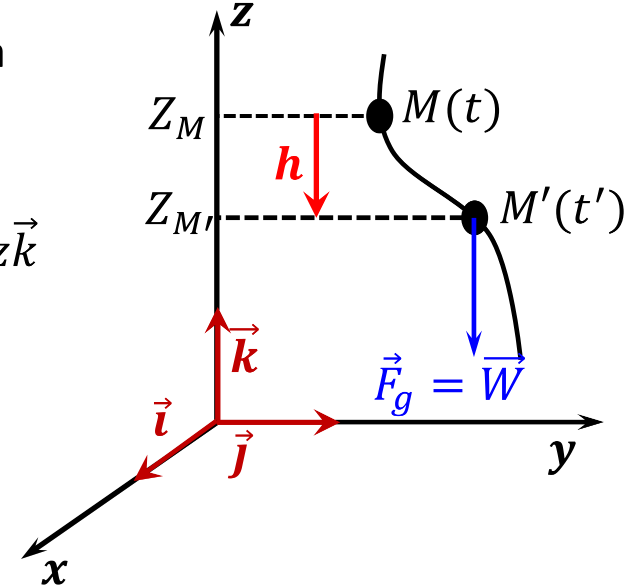
$$dW_{\vec{F}} = \vec{F}.\overrightarrow{dl} \quad \Longrightarrow W_{\vec{F}} = \int_A^B \vec{F}.\overrightarrow{dl}$$

In Cartesian coordinates:

$$\begin{cases} \vec{F} = F_x\vec{\imath} + F_y\vec{\jmath} + F_z\vec{k} \\ \overrightarrow{dl} = dx\vec{\imath} + dy\vec{\jmath} + dz\vec{\jmath} \end{cases} \Longrightarrow \boldsymbol{dW = F_x dx + F_y dy + F_z dz}$$

III.2.3. Work Done By Gravitational Force:

Gravitational force $\vec{F_g}$ is the force that keeps anything with a mass m attracted to the earth.

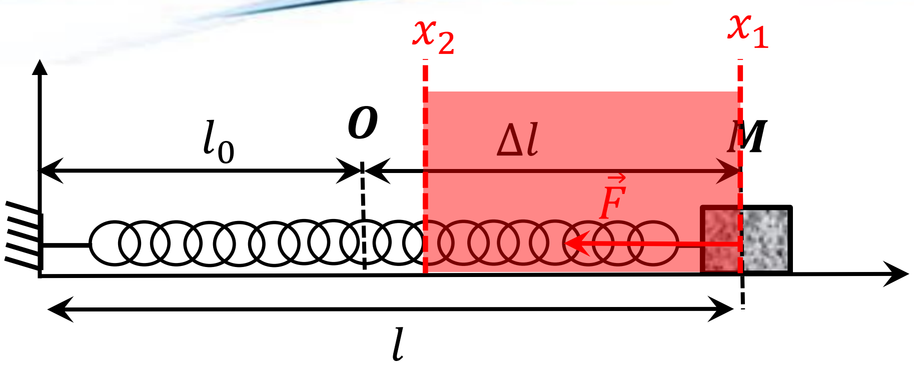


$$W_{\vec{F_g}} = \int_M^{M'} \vec{F_g}.\overrightarrow{dl}\,, \quad \vec{F_g} = -mg\vec{k}\,, \quad \overrightarrow{dl} = dx\vec{\imath} + dy\vec{\jmath} + dz\vec{k}$$

$$\Longrightarrow W_{\vec{F_g}} = \int_M^{M'} -mg.dz \quad = -mg(Z_{M'} - Z_M)$$

Either : $h = Z_M - Z_{M'}$ $\boxed{\Longrightarrow \boldsymbol{W_{\vec{F_g}} = mgh}}$

III.2.4. Work done by an elastic force:



$$W = \int \vec{F}.\overrightarrow{dl}$$

We have :

- $\vec{F} = -k\overrightarrow{OM} = -k(l - l_0)\vec{\imath} = -kx\vec{\imath}$
- $\overrightarrow{dl} = dx\vec{\imath}$

$$\Longrightarrow W = \int -kx\vec{\imath}.dx\vec{\imath} = -k\int xdx$$

$$\boldsymbol{\Longrightarrow W = -\frac{1}{2}kx^2 + Cts}$$

When $\vec{F}$ moves from the x_1 position to x_2 position , We have :

$$\boldsymbol{W = -k\int_{x_1}^{x_2} xdx = -\frac{1}{2}k\left({x_2}^2 - {x_1}^2\right)}$$

The work of this force does not depend on the path followed but only on the initial and final position of the spring

III.2.5. Power of Force:

Power is the rate at which work is done or energy is transferred in a unit of time.

- Average Power: $P_{ave} = \dfrac{\Delta W_{\vec{F}}}{\Delta t}$
- Instantaneous Power: $P_{inst} = P(t) = \dfrac{dW_{\vec{F}}}{dt}$

$([\boldsymbol{P}] = \boldsymbol{Watt})$

The power of a force $\vec{F}$ in a time interval dt manages to move a mobile by a distance dl can be written by:

$$P(t) = \frac{dW_{\vec{F}}}{dt} = \frac{\vec{F}.\overrightarrow{dl}}{dt} = \vec{F}.\frac{\overrightarrow{dl}}{dt} = \vec{F}.\vec{V}$$

III.3. Energy

Energy, in physics , is the capacity for doing work. Energy can neither be created nor destroyed, and it can only be transformed from one form to another.

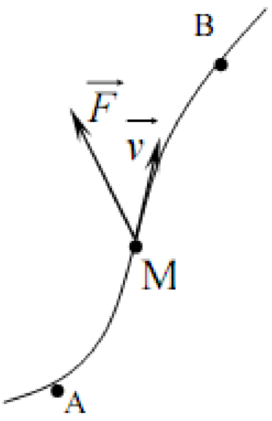
- **Types of Energy**
  - Mechanical energy
  - Chemical energy
  - Electric energy
  - Magnetic energy
  - Radiant energy
  - Nuclear energy
  - Ionization energy
  - Elastic energy
  - Gravitational energy
  - Thermal energy
  - Heat Energy

- All forms of energy are either kinetic or potential:
  - ✓ The energy in motion is known as Kinetic Energy.
  - ✓ Whereas Potential Energy is the energy stored in an object and is measured by the amount of work done.

III. 3.1. Kinetic energy

We define the kinetic energy of a material point M, of mass m and animated with a velocity V , by the quantity Ec , such that :

$$E_C = \frac{1}{2} m V^2$$



- Let a material point $\boldsymbol{M}$, of mass m, moves between points A and B under the action of an external force $\vec{F}$.
- According to the fundamental principle of dynamics, we have:

$$\sum \vec{F}_{ext} = m\vec{a} \Longrightarrow \vec{F} = m \frac{\overrightarrow{dV}}{dt}$$

The elementary work of $\vec{F}$ is given by:

$$dW_{\vec{F}} = \underbrace{\vec{F}}_{m\frac{\overrightarrow{dV}}{dt}} . \underbrace{\overrightarrow{dl}}_{\vec{V}dt} = m \frac{\overrightarrow{dV}}{dt} . \vec{V} dt \quad \left(\text{car } \vec{V} = \frac{\overrightarrow{dl}}{dt} \Longrightarrow \overrightarrow{dl} = \vec{V} dt \right)$$

$$\Longrightarrow dW_{\vec{F}} = m\frac{\overrightarrow{dV}}{dt}.\vec{V}dt = mVdV = d\left(\frac{1}{2}mV^2\right) = dE_C$$

So the work done between A and B is given by:

$$W_{\vec{F}} = \int_A^B \vec{F}.\overrightarrow{dl} = \int_A^B dE_C = E_C(B) - E_C(A)$$

Kinetic energy theorem:

In a Galilean frame of reference, the change in kinetic energy of a material point subjected to a set of external forces between a position A and another position B is equal to the sum of the works of these forces between these two points:

$$\boldsymbol{\Delta E_C = E_C(B) - E_C(A) = \sum W_{A\to B}\left(\vec{F}_{ext}\right)}$$

III.3.2. Conservative and non-conservative forces:

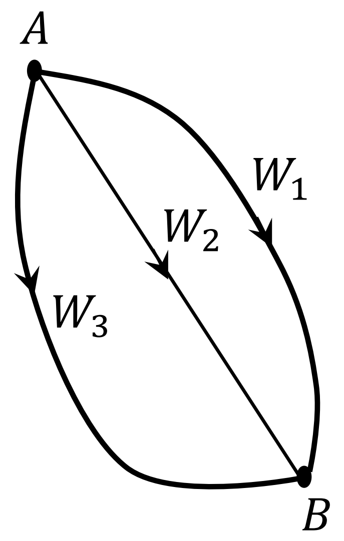
- Forces are said to be conservative when:

  1- Their work does not depend on the path followed but only on the point of departure and the point of arrival.

  For example:

  according to the figure on the right:

  $$W_1(A \to B) = W_2(A \to B) = W_3(A \to B)$$



  2- The total work on a closed path (i.e. a round trip) is zero.

  $$W(A \to A) = W_1(A \to B) + W_3(B \to A) = 0$$

  Examples of conservative forces:

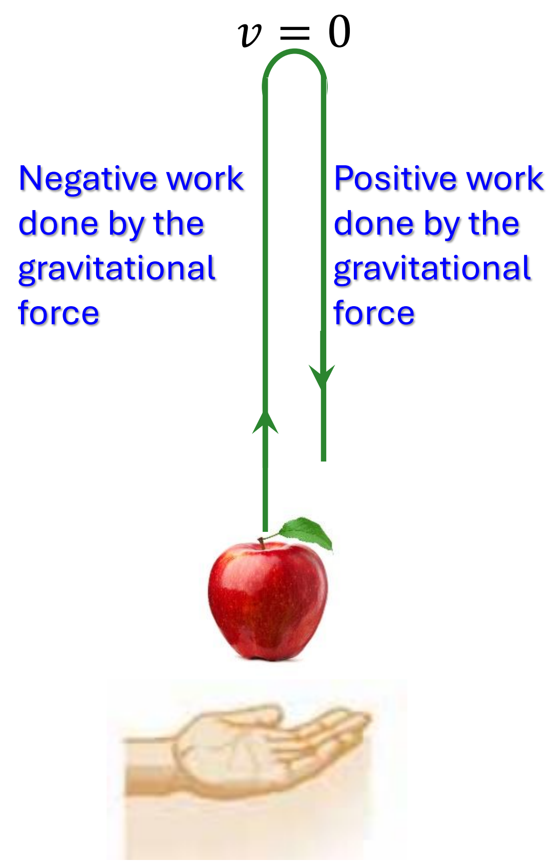
  Gravitational forces, elastic forces, gravitational forces......

- Forces are said to be non-conservative when their work depends on the path taken.

  Example of non-conservative forces: Frictional forces.

III. 3.1. Potential Energy:

The potential energy of a body or physical system is the energy that is present in it and has the potential to transform into kinetic energy.



- Consider an object near the earth's surface as a system with initially upward velocity.
- Once the object is released, the gravitational force, acting as an external force, does a negative amount of work on the object, and the kinetic energy decreases until the object reaches its highest point, at which its kinetic energy is zero.
- The gravitational force then does a positive job until the object returns to its original starting point with a downward velocity.
- <u>If we ignore the effects of air resistance</u>, then the descending object will have the same kinetic energy as when it was launched.
- All kinetic energy has been completely recovered

$\Longrightarrow$ We define the potential energy E_P as the quantity of energy that must be added to the kinetic energy E_C so that their sum is constant:

$$E_C + E_P = Cte$$

- For a displacement producing a change in kinetic energy ΔE_C, the corresponding change in potential energy ΔE_P can be given by:

$$\Delta E_P = E_P(B) - E_P(A) = -\Delta E_C \;\; = -W_{\vec{F}_C}(A \to B)$$

With $\vec{F}_C$ is a conservative force

$$\Longrightarrow \Delta E_P = -\int_A^B \vec{F}_C . \overrightarrow{dl}$$

Using the notion of elementary work dW of a conservative force $\vec{F}_C$:

$$dW = \vec{F}_C . \overrightarrow{dl} \quad \Longrightarrow dE_P = -\vec{F}_C . \overrightarrow{dl}$$

We have:

1- $\begin{cases} \vec{F}_C = F_x\vec{\imath} + F_y\vec{\jmath} + F_z\vec{k} \\ \overrightarrow{dl} = dx\vec{\imath} + dy\vec{\jmath} + dz\vec{k} \end{cases} \quad \Longrightarrow \vec{F}_C . \overrightarrow{dl} = F_x dx + F_y dy + F_z dz$

$2 - \; dE_P = \dfrac{\partial E_P}{\partial x}dx + \dfrac{\partial E_P}{\partial y}dy + \dfrac{\partial E_P}{\partial z}dz$ (Total differential of a function)

$$dE_P = -\vec{F}_C . \overrightarrow{dl} \;\Longrightarrow \frac{\partial E_P}{\partial x}dx + \frac{\partial E_P}{\partial y}dy + \frac{\partial E_P}{\partial z}dz = -F_x dx - F_y dy - F_z dz$$

$$\Longrightarrow \begin{cases} F_x = -\dfrac{\partial E_P}{\partial x} \\ F_y = -\dfrac{\partial E_P}{\partial y} \\ F_z = -\dfrac{\partial E_P}{\partial z} \end{cases} \quad \Longrightarrow F_x\vec{\imath} + F_y\vec{\jmath} + F_z\vec{k} = -\frac{\partial E_P}{\partial x}\vec{\imath} - \frac{\partial E_P}{\partial y}\vec{\jmath} - \frac{\partial E_P}{\partial z}\vec{k} \quad \boldsymbol{= -\vec{\nabla} E_P}$$

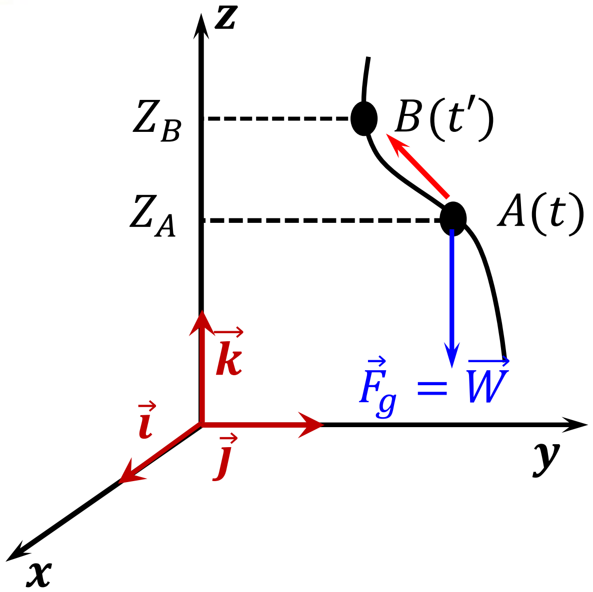
$$\boxed{\Longrightarrow \boldsymbol{\vec{F}_C = -\overrightarrow{grad}E_P}}$$

III. 3.1.1. Potential Energy of the Force of Gravity:

$$\Delta E_P = E_P(B) - E_P(A) = -\int_A^B \vec{F_g}.\overrightarrow{dl}$$

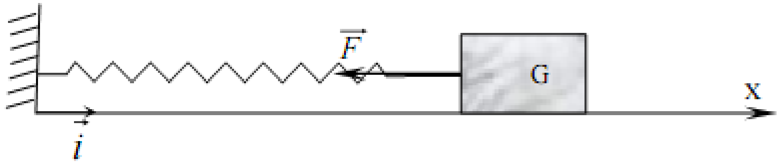
$$\vec{F_g} = \overrightarrow{W} = -mg\overrightarrow{k}\,;\ \overrightarrow{dl} = dx\vec{\imath} + dy\vec{\jmath} + dz\vec{k}$$

$$\Delta E_P = \int_A^B mgdz = mg(Z_B - Z_A) = mgh$$



III. 3.1.2. Potential Energy of an Elastic Force:

$$\vec{F} = -kx\vec{\imath}$$

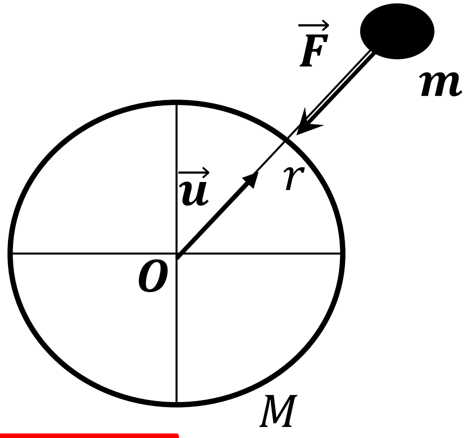


$$\vec{F} = -\overrightarrow{grad}E_P = -\frac{\partial E_P}{\partial x}\vec{\imath} \quad \Longrightarrow \frac{dE_P}{dx} = kx$$

$$E_P = \int kxdx = \frac{1}{2}kx^2 + Cte$$

III. 3.1.3. Potential Energy of a Gravitational Force:

$$\vec{F}(r) = -\frac{GMm}{r^2}\vec{u} = -\frac{GMm}{r^3}\vec{r} \qquad \left(\boldsymbol{\vec{u} = \frac{\vec{r}}{r}}\right)$$



$$\vec{F}(r) = -\overrightarrow{grad}E_P(r) = -\frac{dE_P(r)}{dr}\vec{u} \Longrightarrow \frac{dE_P(r)}{dr} = \frac{GMm}{r^2}$$

$$\Longrightarrow dE_P(r) = \frac{GMm}{r^2}dr \quad \boxed{\boldsymbol{\Longrightarrow E_P(r) = \int \frac{GMm}{r^2}dr = -\frac{GMm}{r} + Cte}}$$

III.3.2. Mechanical energy

Let be a system moving between points A and B under the effect of conservative and non-conservative forces. According to the kinetic energy theorem, we have:

$$E_C(B) - E_C(A) = \sum W_{A\to B}(\vec{F}_C) + \sum W_{A\to B}(\vec{F}_{NC})$$

With :$\vec{F}_C$: Conservative force and $\vec{F}_{NC}$: non-conservative force

We have: $\sum W_{A\to B}(\vec{F}_C) = -(E_P(B) - E_P(A))$

$$\Longrightarrow E_C\ (B) - E_C(A) = -(E_P(B) - E_P(A)) + \sum W_{A\to B}(\vec{F}_{NC})$$

$$\Longrightarrow (E_C(B) + E_P(B)) - (E_C(A) + E_P(A)) = \sum W_{A\to B}(\vec{F}_{NC})$$

- $\boldsymbol{E_C + E_P = E}$: **Called « Machanical energy (Totale)**

$$\boxed{\Longrightarrow \boldsymbol{E(B) - E(A) = \sum W_{A\to B}(\vec{F}_{NC})}}$$

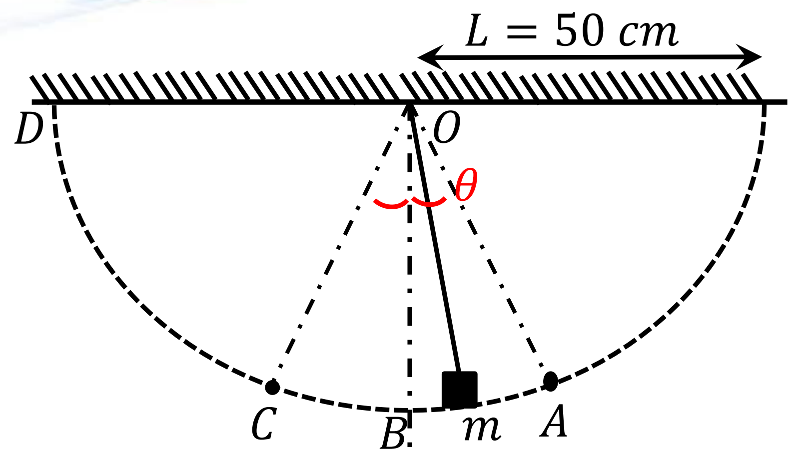
Mechanical Energy Theorem:

The change in the mechanical energy of a system, moving between two points A and B, is equal to the sum of the works of the non-conservative external forces applied to that system :

$$\boldsymbol{E(B) - E(A) = \sum W_{A\to B}\left(\vec{F}_{NC}\right)}$$

However, when the system is isolated (i.e., it is not subject to any non-conservative external forces) the mechanical energy is conserved $\Longrightarrow \Delta \boldsymbol{E} = \boldsymbol{0}$.

Exercise:



A small object of mass m modeled by a point is hung at the end of an inextensible thread of length L. The other end is attached to a bracket(see figure). We do the study in the terrestrial frame of reference. The initial angle is$\theta = 20°$, Length L = 50 cm.

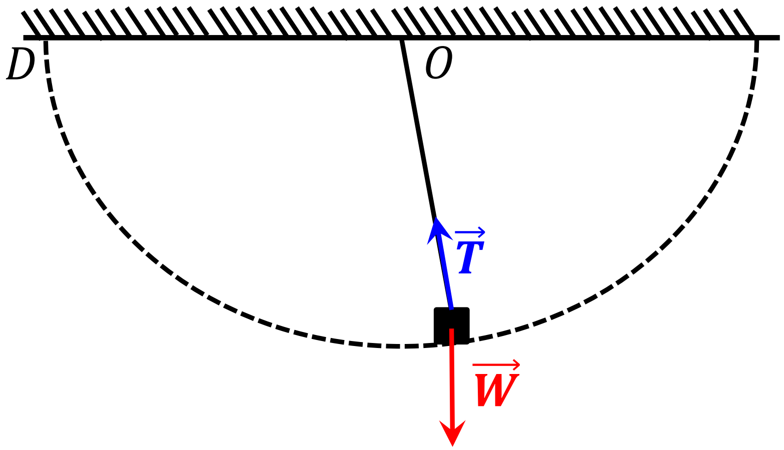
a. Trace the forces acting on the object.

b. We let go of the object from point A. Using the kinetic energy theorem, express its velocity V_B at point B as a function of g, L, and θ , and then calculate it.

c. What is its velocity at point C?

d. We now throw the object from point A with speed $\vec{V}_A$ tangent to the circle, towards the left. Express the minimum value of the norm of V_A for the object to go to point D as a function of g, L and θ. Calculate it.

Solution:



a. the forces acting on the object are:
Object Weight $\vec{W}$ and Thread tension $\vec{T}$

b. In A, the velocity being zero $E_C(A) = 0\ J$.

In B The kinetic energy is $E_c(B) = \frac{1}{2}mV_B{}^2$

Applying the kinetic energy theorem

$$\Delta \boldsymbol{E_C} = \boldsymbol{E_C}(\boldsymbol{B}) - \boldsymbol{E_C}(\boldsymbol{A}) = \sum \boldsymbol{W_{A\to B}}\left(\vec{\boldsymbol{F}}_{\boldsymbol{ext}}\right) = \boldsymbol{W_{A\to B}}\left(\vec{\boldsymbol{T}}\right) + \boldsymbol{W_{A\to B}}\left(\vec{\boldsymbol{W}}\right)$$

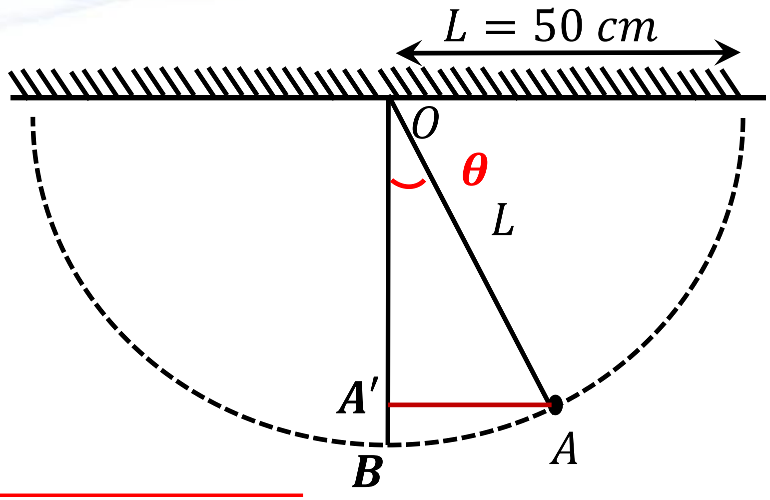
- The tension of the wire $\vec{T}$ is perpendicular to the trajectory, its work is always zero.
- The weight $\vec{W}$ is a conservative force, its work depends only on the start and end positions, and therefore on the difference in altitude h between point A and point B.

$$h = A'B = OB - OA' = L - L\cos\theta$$

$$\Longrightarrow h = L(1 - \cos\theta)$$

$$\Delta E_C = E_C(B) - 0 = W_{A\to B}(\vec{P})$$

$$\Longrightarrow \frac{1}{2}mV_B{}^2 = mgh = mgL(1 - \cos\theta)$$



$$\Longrightarrow V_B = \sqrt{2gL(1 - \cos\theta)} = \sqrt{2.9,8.0,5(1 - \cos 20°)} = 0,77 m/s$$

C. Point C is at the same height as point A, if no energy is lost, the object

is in C with zero velocity, all the mechanical energy is grouped in the potential energy.

D. the object reaches point D with zero velocity

Applying the principle of conservation of mechanical energy between A and D

$$\Delta E_m = 0 \Longrightarrow E_m(D) - E_m(A) = 0$$

$$\Longrightarrow \left(E_C(D) + E_P(D)\right) - \left(E_C(A) + E_P(A)\right) = 0$$

$$\begin{cases} E_C(D) = 0 \\ E_P(D) = mgL \\ E_C(A) = \frac{1}{2} m{V_A}^2 \\ E_P(A) = mgL(1 - cos\theta) \end{cases}$$

$$\Longrightarrow mgL = \frac{1}{2} m{V_A}^2 + mgL(1 - cos\theta) = \frac{1}{2} m{V_A}^2 + mgL - mgLcos\theta$$

$$\boxed{\Longrightarrow \boldsymbol{V_A = \sqrt{2gLcos\theta} = 3\ m/s}}$$