Chapter 5 Linear algebra

1 Algebraic Structures

1.1 Internal Composition Laws

Definition 1.1 *Let E be a set. The function*

$$T: \ E \times E \longrightarrow E$$

$$(x, y) \longmapsto T(x, y)$$

Is called an internal composition law on E (i.c.l)

Remark 1.1 1. *There exist other notations for internal composition laws*

$$\star, \perp, \triangle, +, \times, \ldots$$

2. *An internal composition law is also called an operation*

3. *To prove that an operation $\star$ is internal in E, one shows that whenever we take any two elements x, y from E, the composition $x \star y$ remains within E.*

Example 1.1 *On $E = \mathbb{Z}$*
Addition

$$+: \ \mathbb{Z} \times \mathbb{Z} \longrightarrow \mathbb{Z}$$

$$(x, y) \longmapsto x + y$$

Multiplication

$$\times: \ \mathbb{Z} \times \mathbb{Z} \longrightarrow \mathbb{Z}$$

$$(x, y) \longmapsto x \times y$$

are two internal composition laws on $\mathbb{Z}$.
Division is not an internal composition law on $\mathbb{Z}$, *It is enough to see, for example, that for* $(x, y) = (3, 2) \in \mathbb{Z} \times \mathbb{Z}$ *on a* $3 \div 2 = \dfrac{3}{2} \notin \mathbb{Z}$.

Example 1.2 *We define on* $\mathbb{Z}^*$ *the law* $\star$ *as follow*

$$\forall\,(x, y) \in \mathbb{Z}^* \times \mathbb{Z}^*,\ x \star y = \frac{x+y}{2}$$

Calculate $1 \star 1,\ 2 \star 3, (-5) \star 5$.
The law $\star$ *Is it internal in* $\mathbb{Z}^*$.

Solution 1.2 *We have* $\forall\,(x, y) \in \mathbb{Z}^* \times \mathbb{Z}^*,\ x \star y = \dfrac{x+y}{2}$. *Therefore*
1) $1 \star 1 = \dfrac{1+1}{2} = 1$
2) $(-5) \star 5 = \dfrac{-5+5}{2} = 0$
3) $2 \star 3 = \dfrac{3+2}{2} = \dfrac{5}{2} \notin \mathbb{Z}^*$
From 3) *we deduce that the law* $\star$*is not internal.*

1.2 Properties of Internal Composition Laws

In all that follows, we assume that E is a set equipped with an internal composition law denoted by $\star$.

Commutativity

We say that $\star$ is commutative if and only if

$$\forall x, y \in E \qquad x \star y = y \star x.$$

Example 1.3 *Let* $\star$ *an internal composition law defined on* $\mathbb{Z}$ *by*

$$x \star y = x + y + 1$$

$\star$ *Is it commutative ?*

Solution 1.3 *Let* $x, y \in \mathbb{Z}$
$x \star y = x + y + 1 = y + x + 1 = y \star x$
So $\star$ *is a commutative internal composition law.*

Remark 1.2 $+$, $\times$ *are two commutative internal composition laws on* $\mathbb{R}$.

Associativity

We say that $\star$ is associative if and only if

$$\forall x, y, z \in E \qquad (x \star y) \star z = x \star (y \star z)\,.$$

Example 1.4 *We define on* $E = [0\ 1]$ *an internal composition law* $\star$ *by*

$$\forall x, y \in E \qquad x \star y = x + y - xy$$

$\star$ *is it associative?*

Solution 1.4 *Let* $x, y, z \in E$

$$\begin{aligned} (x \star y) \star z &= (x + y - xy) \star z \\ &= (x + y - xy) + z - (x + y - xy)\, z \\ &= x + y - xy + z - xz - yz + xyz. \\ &= x + y + z - xy - xz - yz + xyz \text{....................}(1) \end{aligned}$$

$$\begin{aligned} x \star (y + z) &= x \star (y + z - yz) \\ &= x + (y + z - yz) - x\,(y + z - yz) \\ &= x + y + z - yz - xy - xz + xyz \\ &= x + y + z - xy - yz - xz + xyz \text{.................}(2) \end{aligned}$$

from (1) *and* (2) *we deduce that* $(x \star y) \star z = x \star (y + z)$, *so* $\star$ *is an associative internal composition law*

Example 1.5 $E = \mathbb{Z}$ $\forall x, y \in E \qquad x \star y = 2x + y + 5$

Solution 1.5 *Let* $x, y, z \in E$

$$
\begin{aligned}
(x \star y) \star z &= (2x + y + 5) \star z \\
&= 2(2x + y + 5) + z + 5 \\
&= 4x + 2y + 10 + z + 5. \\
&= 4x + 2y + z + 15 \text{..................}(1) \\
x \star (y + z) &= x \star (2y + z + 5) \\
&= 2x + (2y + z + 5) + 5 \\
&= 2x + 2y + z + 5 + 5 \\
&= 2x + 2y + 10 \text{..............}(2)
\end{aligned}
$$

from (1) et (2) we deduce that $(x \star y) \star z \neq x \star (y + z)$. As we can see $\star$ is not associative.

Identity Element

Let $e \in E$, we say that e is an identity element in E for the operation $\star$ iff

$$\forall x \in E \qquad x \star e = e \star x = x.$$

Example 1.6 *0 is the identity element for the addition in* $\mathbb{R}$
1 is the identity element for the multiplication in $\mathbb{R}$

Example 1.7 *We define on* $E =]-1\ 1[$ *the operation* $\top$ *as follow*

$$\forall x, y \in E \qquad x \top y = \frac{x + y}{1 + xy}$$

Find the identity element in E *for the operation* $\top$.

Solution 1.6 *Let* $e \in E$ *such that* $\quad \forall x \in E \;\; x \top e = e \top = x$

$$
\begin{aligned}
x \top e = x &\iff \frac{x + e}{1 + xe} = x \\
&\iff x + e = x(1 + xe) \\
&\iff x + e = x + x^2 e \\
&\iff e - x^2 e = 0 \\
&\iff e\left(1 - x^2\right) = 0 \\
&\implies e = 0 \textit{ because } \left(1 - x^2\right) \neq 0 \textit{ for } x \in]-1\ 1[
\end{aligned}
$$

In the same way, for $e \top x = x$, we find $e = 0$. Therefore, 0 is the identity element in $]-1;1[$ for the operation $\top$.

Remark 1.3 *The identity element, if it exists, is unique.*

Invertible Element

We assume that the operation $\star$ has an identity element in E denoted as e. For an element $x \in E$, we say that the element $x' \in E$ is the symmetric or the inverse of x in E with respect to the operation $\star$ if and only if

$$x \star x' = x' \star x = e.$$

If x is invertible then its inverse is denoted x^{-1}.

Example 1.8 *$-x$ is the symmetric of x with respect to addition in $\mathbb{R}$.*
Let $x \neq 0$, the symmetric of x with respect to multiplication in $\mathbb{R}$ is $\dfrac{1}{x}$.

Example 1.9 *Let $\triangle$ an internal composition law defined on $\mathbb{Z}$ by*

$$\forall x, y \in \mathbb{Z} \quad x \triangle y = 2x - y + 1$$

Find the symmetric, if it exists, of an element x in $\mathbb{Z}$.

Solution 1.7 *Let $x \in \mathbb{Z}$, before looking for the symmetric of x dans $\mathbb{Z}$ We must check if the identity element for the operation $\triangle$ exists in $\mathbb{Z}$.*
Let $e \in \mathbb{Z}$, in order that e be an identity element, it is necessary and sufficient that it satisfies:

$$x \triangle e = e \triangle x = x \quad \textit{where} \quad x \in \mathbb{Z}.$$

$(x \triangle e = x) \Longrightarrow (e = x + 1)$. So, it is clear that e does not exist, since according to this result, for each value of x, we have a value of e, whereas the identity element, if it exists, is unique
Since the identity element for the operation $\triangle$ does not exist, the elements of $\mathbb{Z}$ are not invertible for the operation $\triangle$.

Example 1.10 *Let $\triangle$ an internal composition law defined on $\mathbb{R}_+^*$ par*

$$\forall x, y \in \mathbb{R}_+^* \quad x \triangle y = \sqrt{x^2 + y^2}$$

Find the symmetric, if it exists, of an element x in $\mathbb{R}_+^*$.

Solution 1.8 *It is clear that* 0 *is the identity element for this operation. Indeed*

$$\forall x \in \mathbb{R}_+^* \quad x \triangle 0 = \sqrt{x^2 + 0^2} = |x| = x = 0 \triangle x.$$

Let $x \in \mathbb{R}_+^*$, *we say that* $x' \in \mathbb{R}_+^*$ *is the symmetric of* x *for the operation triangle if and only if*

$$x \triangle x' = 0 = x' \triangle x.$$

$x \triangle x' = 0 \Longrightarrow \sqrt{x^2 + x'^2} = 0$ *et* $x' \triangle x = 0 \Longrightarrow \sqrt{x'^2 + x^2} = 0.$
It follows that $x^2 + x'^2 = 0$, *which is impossible since* x, x' *are two strictly positive elements. Therefore, no element of* $\mathbb{R}_+^*$ *has a symmetric for* $\triangle$.

Remark 1.4

- *If* e *is the identity element in* E *for the operation* $\star$, *then* e *is invertible, and* $e^{-1} = e$.
- *If* x, y *are invertible for the operation* $\star$, *then* $x \star y$ *is invertible, and we have* $(x \star y)^{-1} = y^{-1} \star x^{-1}$.
- *If* a *is invertible for the operation* $\star$, *then the equation* $a \star x = b$ *has a solution* $x = a^{-1} \star b$. *It is easy to see*

$$\begin{aligned} a \star x = b &\iff a^{-1} \star a \star x = a^{-1} \star b \\ &\iff e \star x = a^{-1} \star b \\ &\iff x = a^{-1} \star b \end{aligned}$$

Distributivité

Now, let's assume that E is equipped with two internal composition laws, $\star$ and $\top$. We say that $\star$ is distributive with respect to $\top$ if and only if

$$\forall x, y, z \in E \quad x \star (y \top z) = (x \star y) \top (x \star z) \, .$$

Example 1.11 *We define on* $\mathbb{Z}$ *Two internal composition laws* $\star$ *and* $\top$ *by*

$$\forall x, y \in \mathbb{Z} \quad x \star y = x + y + 3$$

et

$$\forall x, y \in \mathbb{Z} \quad x \top y = xy$$

$\star$ *is it distributive with respect to* $\top$*?*
$\top$ *is it distributive with respect to* $\star$*?*

Solution 1.9 *Let* $x, y, z \in \mathbb{Z}$

$$x \star (y \top z) = x + (y \top z) + 3 = x + yz + 3 \quad ....(1)$$

$$\begin{aligned}(x \star y) \top (x \star z) &= (x \star y) \times (x \star z) \\ &= (x + y + 3)(x + z + 3). \quad ....(2)\end{aligned}$$

Since $(1) \neq (2)$ *then* $\star$ *is not distributive with respect to* $\top$.

Now, we interchange the positions of the two operations, and we obtain,

$$x \top (y \star z) = x \times (y \star z) = x \times (y + z + 3) = xy + xz + 3x \quad ....(1)$$

$$(x \top y) \star (x \top z) = (x \top y) + (x \top z) + 3 = xy + xz + 3. \quad ....(2)$$

Since $(1) = (2)$ *then* $\top$ *is distributive with respect to* $\star$.

1.3 Group Structure

$(E, \star)$ is called a groupe iff

1. $\star$ is associative
2. $\star$ admit an identity element
3. every element in E admits a symetric element in E.

If, moreover, $\star$ is commutative, then $(E, \star)$ is a **commutative** or **Abelian group**.

Example 1.12 *1.* $(\mathbb{Z}, +)$ *is a commutative group.*

2. $(\mathbb{R}, \times)$ *is not a group because* 0 *does not admit a symmetrical element.*

3. $\left(\mathbb{R}_+^*, \times\right)$ *is an Abelian group.*

Subgroup

Let $(E, \star)$ a group and $F \subset E$. We say that F is a subgroup of $(E, \star)$ iff

1. F is stable under the operation $\star$, e.i., $\forall x, y \in F \quad x \star y \in F$
2. $(F, \star)$ is itself a group.

Example 1.13 $(\mathbb{R}, \times)$ *is a group* $\left(\mathbb{R}_{+}^{*}, \times\right)$ *Is a subgroup of this group.*

Characterization of a subgroup

Let $(E, \star)$ a group and $F \subset E$, We have the following equivalence

$$F \text{ est un sous groupe de } \quad (E, \star) \Longleftrightarrow \begin{cases} F \neq \emptyset \\ \forall x \in F \;\; x^{-1} \in F \\ \forall x, y \in F \;\; x \star y \in F \end{cases}$$

Proof. We prove the following two implications
$\Longrightarrow$) If F is a subgroup then it is itself a group, so we deduce that F is not empty because it contains the identity element. Moreover, the inverse of each element of F belongs to F. Furthermore, the stability of the operation gives us that for all $x, y \in F$ we have $x \star y \in F$.

$\Longleftarrow$) We assume that we have

$$\begin{cases} (1) \quad F \neq \emptyset \\ (2) \quad \forall x \in F \;\; x^{-1} \in F \\ (3) \quad \forall x, y \in F \;\; x \star y \in F \end{cases}$$

1) F Is stable under the operation $\star$ according to the assumption (3).
2) $(F, \star)$ Is itself a group because:

- $\forall x \in F,\ x^{-1} \in F$ and $x \star x^{-1} \in F$ then $e \in F$
- $\star$ is associative in E, so it is associative in F $(F \subset E)$

from 1) and 2) we deduce that F is a subgroup of $(E, \star)$. ∎

Example 1.14 *Soit* $(E, \star)$ *un groupe non commutatif et* F *une partie de* E *telle-que*

$$F = \left\{ a \in E : \quad a \star x = x \star a \quad \forall x \in E \right\}$$

Montrer que F *est un sous groupe de* $(E, \star)$.

Solution 1.10 1) $\quad F \neq \emptyset$
Indeed, let e be the identity element in the group $(E, \star)$, we have

$$\forall x, y \in E \;\; x \star e = e \star x$$

$$\textit{then} \quad e \in F$$

2) $\quad F$ *is stable by* $\star$
Indeed, let $a, b \in F$ then we have

$$x \star a = a \star x \quad \textit{et} \quad x \star b = b \star x, \;\; \forall x \in E$$

we have to prove that $a \star b \in F$, e.i., $x \star (a \star b) = (a \star b) \star x \quad \forall x \in E$.
Since $a, b \in F \subset E$ and since $\star$ is associative on E, we have for all $x \in E$

$$\begin{aligned} x \star (a \star b) &= (x \star a) \star b && \textit{by associativity} \\ &= (a \star x) \star b && \textit{beacause } a \in F \\ &= a \star (x \star b) && \textit{by associativity} \\ &= a \star (b \star x) && \textit{because } b \in F \\ &= (a \star b) \star x && \textit{by associativity} \end{aligned}$$

donc $(a \star b) \in F$.
3) *let $a \in F$ we have*

$$x \star a = a \star x \quad \forall x \in E$$

We want to prove that $a^{-1} \in F$, e.i.,

$$x \star a^{-1} = a^{-1} \star x \quad \forall x \in E.$$

Let e The identity element in E, since $F \subset E$ it yieds $a \in E$ and therefore the inverse of a exists in E and satisfies

$$a \star a^{-1} = a^{-1} \star a = e$$

Let $x \in E$, We know that $\star$ is an internal composition law in E then $x \star a^{-1} \in E$, Moreover, we have

$$\left(x \star a^{-1}\right) \star e = e \star \left(x \star a^{-1}\right) = \left(x \star a^{-1}\right),$$

Therefore, we have

$$\begin{aligned}
x \star a^{-1} &= e \star \left(x \star a^{-1}\right) \\
&= \left(a^{-1} \star a\right) \star \left(x \star a^{-1}\right) \\
&= a^{-1} \star a \star x \star a^{-1} \\
&= a^{-1} \star (a \star x) \star a^{-1} \quad \textit{by associativity} \\
&= a^{-1} \star (x \star a) \star a^{-1} \quad \textit{because } a \in F \\
&= \left(a^{-1} \star x\right) \star \left(a \star a^{-1}\right) \quad \textit{by associativity} \\
&= a^{-1} \star x
\end{aligned}$$

so $a^{-1} \in F$.
Conclusion: According to 1), 2), 3), *we deduce that* F *is a subgroup of* $(E, \star)$.

Morphisme de groupe

Let $(E, \star)$ and $(H, \top)$ two groups a $f: \ E \longrightarrow H$ a function.

- We say that f is a group homomorphism if and only if

$$\forall x, y \in E \quad f(x \star y) = f(x) \top f(y).$$

- If, in addition, f is bijective, we refer to it as an isomorphism of groups.
- If $E = H$ et $\star = \top$ then f is called endomorphism of groups.
- If f is a bijective endomorphism, it's called an automorphism.

Example 1.15 *Consider the function*

$$f: \ \mathbb{R} \longrightarrow \mathbb{R}^*$$

$$x \longmapsto f(x) = e^x$$

Show that f *is a group homomorphism from* $(\mathbb{R}, +)$ *to* $(\mathbb{R}^*, \times)$

Solution 1.11 *Let* $x, y \in \mathbb{R}$

$$f(x+y) = e^{x+y} = e^x e^y = f(x) \times f(y) \quad \textit{the proof is complete}$$

Example 1.16 *We consider the function*

$$f: \mathbb{R}^* \longrightarrow \mathbb{R}^*$$
$$x \longmapsto f(x) = x^n \quad n \in \mathbb{N}$$

f is it an endomorphisme on $(\mathbb{R}^*, \cdot)$?

Solution 1.12 *let* $x, y \in \mathbb{R}^*$

$$f(x.y) = (x.y)^n = x^n.y^n = f(x).f(y) \quad \textit{the proof is complete}$$

1.4 Ring structure

Let E a set equipped with two internal composition laws $\star$ et $\top$.
We say that $(E, \star, \top)$ is a ring if and only if

1. $(E, \star)$ is an Abelian group,
2. the law $\top$ is associative,
3. the law $\top$ is distributive with respect to $\star$ on the left and on the right. That is

   $$\forall x, y, z \in E \quad x\top(y \star z) = (x\top y)\star(x\top z) \;\text{ et }\; (x \star y)\top z = (x\top z)\star(y\top z).$$

If, in addition, the operation $\top$ is commutative, then the ring $(E, \star, \top)$ is commutative.
If the neutral element with respect to the operation $\top$ exists in E, then the ring $(E, \star, \top)$ is called unitary.

Example 1.17 $(\mathbb{Z}, +, \times)$ *is a commutatif ring.*
$(\mathbb{Z}, \times, +)$ *is not a ring.*

1.5 Field structure

Let E be a set equipped with two internal composition laws $\star$ and $\top$.
$(E, \star, \top)$ is called a field if and only if

1. $(E, \star, \top)$ is a unitary ring,
2. every element of $E - \{e\}$ is invertible, where e is the neutral element with respect to the operation $\star$.

If, in addition, the operation $\top$ is commutative, then the field $(E, \star, \top)$ is commutative.

Example 1.18 $(\mathbb{R}, +, \times)$ *is a commutative field.*

2 Vector Spaces- Sub Vector Spaces

2.1 Vector Spaces

Let $\mathbb{K}$ be a commutative field (generally $\mathbb{R}$ or $\mathbb{C}$) and let E be a set equipped with an internal operation denoted by $(+)$ and an external operation denoted by $(.)$, such that

$$\begin{aligned}(+):\quad & E\times E \longrightarrow E\\ &(x,y)\longmapsto x+y\end{aligned}$$

$$\begin{aligned}(.):\quad & \mathbb{K}\times E \longrightarrow E\\ &(\lambda,x)\longmapsto \lambda.x\end{aligned}$$

Definition 2.1 *$(E,+,.)$ is a vector Space on $\mathbb{K}$ or a $\mathbb{K}-$vector Space If the following properties are satisfied:*

1. *$(E,+)$ is an Abelian groupe ,*
2. *$\forall\lambda\in\mathbb{K},\forall x,y\in E \quad \lambda.(x+y)=\lambda.x+\lambda.y$,*
3. *$\forall\lambda,\mu\in\mathbb{K},\forall x\in E \quad (\lambda+\mu).x=\lambda.x+\mu.x$,*
4. *$\forall\lambda,\mu\in\mathbb{K},\forall x\in E \quad \lambda.(\mu.x)\,(\lambda\mu).x=(\lambda.x).\mu$,*
5. *$\forall x\in E \quad 1_{\mathbb{K}}.x=x$*

If E is a $\mathbb{K}$-vector space, then the elements of E are called vectors, and those of $\mathbb{K}$ are called scalars

Remark 2.1 *To simplify notations, we write λx instead of $\lambda.x$*

Properties

Let E be a $\mathbb{K}$-vector space, we have the following properties

1. $\forall\lambda\in\mathbb{K},\forall x\in E \quad \left[\lambda x=0_E \Longleftrightarrow (\lambda=0_{\mathbb{K}}\vee x=0_E)\right]$,
2. $\forall\lambda\in\mathbb{K},\forall x\in E \quad \lambda(-x)=-\lambda x$,
3. $\forall\lambda\in\mathbb{K},\forall x,y\in E \quad \lambda(x-y)=\lambda x-\lambda y$,
4. $\forall x\in E, \quad 0_{\mathbb{K}}.x=0_E$,
5. $\forall\lambda\in\mathbb{K}, \quad \lambda.0_E=0_E$

2.2 Sub Vector Spaces

Let $(E,+,.)$ be a vector space, and $F \subset E$. We say that F is a sub-vectorial-space (s.v.s.) of E if one of the following equivalent properties is satisfied:

1. $(F,+,.)$ a vector space.
2. $\begin{cases} F \neq \emptyset, \\ \forall x, y \in F, \forall \lambda, \mu \in \mathbb{K} \quad (\lambda x + \mu y) \in F. \end{cases}$
3. $\begin{cases} F \neq \emptyset \\ \forall x, y \in F \quad x + y \in F \\ \forall \lambda \in \mathbb{K}, \forall x \in F \quad \lambda x \in F. \end{cases}$

Remark 2.2 *This remark is very useful in practice.*

- *To show that $F \neq \emptyset$, it is sufficient to prove that $0_E \in F$.*
- *If $0_E \notin F$ then F cannot be a vector subspace.*

Example 2.1 *Set $E = \mathbb{R}^2$ and $F = \left\{ (x,y) \in \mathbb{R}^2 \mid y = 2x \right\}$*
Show that F is a vector subspace of $\mathbb{R}^2$.

Solution 2.2 *We prove that F satisfies* $\begin{cases} F \neq \emptyset \\ \forall x, y \in F \quad x + y \in F \\ \forall \lambda \in \mathbb{K}, \forall x \in F \quad \lambda x \in F. \end{cases}$

1. $(0,0) \in F$ *because* $\quad 0 = 2 \times 0,$

$$\textit{then } F \neq \emptyset.$$

2. Let $X = (x_1, y_1)$ et $Y = (x_2, y_2)$ two elements of F, e.i.,

$$y_1 = 2x_1 \ \textit{ et } \ y_2 = 2x_2.$$

We have

$$X + Y = (x_1 + x_2, y_1 + y_2) \ \textit{ et } \ y_1 + y_2 = 2x_1 + 2x_2 = 2(x_1 + x_2),$$

$$\textit{then } X + Y \in F.$$

3. *Let $\lambda \in \mathbb{K}$ and $X = (x_1, y_1) \in F$.*

$$(x_1, y_1) \in F \Longleftrightarrow y_1 = 2x_1.$$

We have

$$\lambda X = (\lambda x_1, \lambda y_1) \quad avec \quad \lambda y_1 = \lambda (2x_1) = 2 (\lambda x_1),$$

$$then \ \lambda X \in F.$$

Conclusion: F is a vector subspace of $\mathbb{R}^2$.

Proposition 2.3 *Let $(E, +, .)$ un $\mathbb{K}-$ Vector Space. If F_1 et F_2 are two vector subspaces of E then $F_1 \cap F_2$ is a vector subspace of E.*

Proof. Let F_1, F_2 two vector subspaces of a $\mathbb{K}-$ vector space E.

1. $0_E \in F_1$ and $0_E \in F_2$ then $0_E \in F_1 \cap F_2$ and consequently

$$F_1 \cap F_2 \neq \emptyset.$$

2. Let $x, y \in F_1 \cap F_2$ then $x, y \in F_1$ et $x, y \in F_2$. Since F_1 and F_2 are v.s.s of E then $x + y \in F_1$ and $x + y \in F_1 \cap F_2$ therefore

$$x + y \in F_1 \cap F_2$$

3. Let $x \in F_1 \cap F_2$ then $x \in F_1$ et $x \in F_2$. Since F_1 et F_2 are v.s.s of E then for all $\lambda \in \mathbb{R}$, we have $\lambda x \in F_1$ and $\lambda x \in F_2$ then

$$\lambda x \in F_1 \cap F_2.$$

Conclusion: $F_1 \cap F_2$ is a v.s.s of E.

■

Remark 2.3 *Generally; the union of two vector spaces of E; is not a vector subspace of E.*

Example 2.2 *We consider the following two vector subspaces*
$F_1 = \Big\{ (x, y) \in \mathbb{R}^2 \ | \ x = 0 \Big\}, \quad F_2 = \Big\{ (x, y) \in \mathbb{R}^2 \ | \ y = 0 \Big\}$
$F_1 \cup F_2$ is it a v.s.s. of $\mathbb{R}^2$?

Solution 2.4 *If $X = (x_1, y_1) \in F_1 \cup F_2$ then $X \in F_1$ where $X \in F_2$. If we consider the two elements $(0, 2), (-3, 0)$ which are in $F_1 \cup F_2$, we have $(0, 2) + (-3, 0) = (-3, 2) \notin F_1 \cup F_2$ because $(-3, 2) \notin F_1$ and $(-3, 2) \notin F_2$, This means that you have found two elements that belong to the union of $F_1 \cup F_2$ but their sum is not in $F_1 \cup F_2$. Then $F_1 \cup F_2$ is not a v.s.s. of $\mathbb{R}^2$.*

2.3 Somme et somme directe

Definition 2.5 *Let E is $\mathbb{K}-$ a vector space and F_1, F_2 two v.s.s of E. The sum F_1 and F_2 is the subset of E denoted $F_1 + F_2$ and which is defined by*

$$F_1 + F_2 = \Big\{\eta \in E \mid \eta = x + y \ \ where \ \ x \in F_1 \ \ and \ \ y \in F_2 \Big\}$$

Proposition 2.6 *$F_1 + F_2$ is a vector subspace of E.*

Definition 2.7 *Let E a $\mathbb{K}-$ vector space and F_1, F_2 two v.s.s of E. We say that F_1 and F_2 are supplementary or that E is the direct sum of F_1 and F_2 if and only if*

$$E = F_1 + F_2 \ \ et \ \ F_1 \cap F_2 = \{0_E\}$$

et on écrit

$$E = F_1 \oplus F_2$$

Proposition 2.8 $\Big(F_1$ *et* F_2 *are supplementary to each other in* $E\Big) \Longleftrightarrow \Big(\forall \eta \in E$ *there exists a unique* $x \in F_1$ *and there exists a unique* $y \in F_2$ *such-that* $\eta = x + y\Big)$

Example 2.3 $E = \mathbb{R}^2$, $F_1 = \Big\{ (x, y) \in \mathbb{R}^2 \ \ | \ \ x = 0\Big\}$, $\quad F_2 = \Big\{ (x, y) \in \mathbb{R}^2 \ | \ \ y = 0\Big\}$

$$E = F_1 \oplus F_2$$

3 Base and dimension

Let $(E, +, .)$ a $\mathbb{K}-$ vector space.

Definition 3.1 *(Linear Combination)*
let $x_1, x_2, x_3, \ldots, x_n$, be n vector of E and $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$, n scalars in $\mathbb{K}$. We call linear combinations of the n vectors of E the following sum

$$\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 + \cdots + \lambda_n x_n.$$

Definition 3.2 *(Generating family)*
We say that the n vectors $x_1, x_2, x_3, \ldots, x_n$ of E Generate E, or that $\{x_1, x_2, x_3, \ldots, x_n\}$ is a Generating family of E iff

$$\forall X \in E, \exists \lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n \in \mathbb{K} \ \ such\text{-}that \ \ X = \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 + \cdots + \lambda_n x_n,$$

and we write $E = \langle x_1, x_2, \ldots, x_n \rangle$ where $E = Vect\,(x_1, x_2, x_3, \ldots, x_n)$.

Example 3.1 *Prove that the vectors* $X = (1, 1), Y = (1, 0)$ *generate* $\mathbb{R}^2$

Solution 3.3 *Let us consider* $Z = (z_1, z_2)$ *we prove that there exists* $\lambda_1, \lambda_2 \in \mathbb{R}$ *such that*

$$Z = \lambda_1 X + \lambda_2 Y$$

$$\begin{aligned} Z = \lambda_1 X + \lambda_2 Y &\iff (z_1, z_2) = \lambda_1 (1, 1) + \lambda_2 (1, 0) \\ &\iff (z_1, z_2) = (\lambda_1, \lambda_1) + (\lambda_2, 0) \\ &\iff (z_1, z_2) = \big(\lambda_1 + \lambda_2, \lambda_1\big) \\ &\Longrightarrow \begin{cases} \lambda_1 + \lambda_2 = z_1 \\ \lambda_1 = z_2 \end{cases} \\ &\Longrightarrow \begin{cases} \lambda_2 = z_1 - z_2 \\ \lambda_1 = z_2 \end{cases} \end{aligned}$$

since z_1, z_2 *are real numbers then* λ_1, λ_2 *exist.*

Example 3.2 $E = \Big\{ (x, y, z) \in \mathbb{R}^3 \mid x + y = 0 \Big\}$
Find a generating family of E.

Solution 3.4 *Let* $X \in E$, *then* $X = (x, y, z)$ *avec* $x + y = 0$. *We have* $x = -y$, *therefore* $X = (-y, y, z) = (-y, y, 0) + (0, 0, z) = y(-1, 1, 0) + z(0, 0, 1) = yx_1 + zx_2$
avec $x_1 = (-1, 1, 0)$ *et* $x_2 = (0, 0, 1)$. *We write*

$$E = \langle (-1, 1, 0), (0, 0, 1) \rangle \quad ou \ E = Vect\Big((-1, 1, 0), (0, 0, 1) \Big)$$

Definition 3.5 *(Linearly independent vectors)*
The n *vectors* $x_1, x_2, x_3, \ldots, x_n$ *of* E *are linearly independent or that the family* $\{x_1, x_2, x_3, \ldots, x_n\}$ *is free if and only if* $\forall \lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n \in \mathbb{K}$,

$$\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 + \cdots + \lambda_n x_n = 0_E \Longrightarrow \lambda_1 = \lambda_2 = \lambda_3 = \cdots = \lambda_n = 0_{\mathbb{K}}$$

if the vectors $x_1, x_2, x_3, \ldots, x_n$ *If they are not linearly independent, then they are called linearly dependent, or the family* $\{x_1, x_2, x_3, \ldots, x_n\}$ *is linearly dependent.*

Example 3.3 *Prove that* $e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (0, 0, 1)$ *are linearly independent.*

Solution 3.6 *let* $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ *such-that* $\lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 = 0_{\mathbb{R}^3}$

$$\begin{aligned}
\lambda_1 (1,0,0) + \lambda_2 (0,1,0) + \lambda_3 (0,0,1) = (0,0,0) &\iff (\lambda_1, 0, 0) + (0, \lambda_2, 0) + (0, 0, \lambda_3) = (0,0,0) \\
&\iff (\lambda_1, \lambda_2, \lambda_3) = (0,0,0) \\
&\Longrightarrow \lambda_1 = \lambda_2 = \lambda_3 = 0. \text{ the proof is complete}
\end{aligned}$$

Definition 3.7 *(Basis of a Vector Space)*
The n vectors $x_1, x_2, x_3, \ldots, x_n$ of E form a basis for E iff the family $\{x_1, x_2, x_3, \ldots, x_n\}$ is a linearly independent and generating family of E.

Definition 3.8 *(Canonical Basis)*
Let $e_1 = (1,0,0,0,\ldots,0)$, $e_2 = (0,1,0,0,\ldots,0)$, $e_3 = (0,0,1,0,\ldots,0)$, $\ldots$, $e_n = (0,0,0,0,\ldots,1)$, n vectors de $\mathbb{R}^n$, $n \in \mathbb{N}$. The vectors $e_1, e_2, e_3, \ldots, e_n$ form a basis for $\mathbb{R}^n$ which is called the canonical basis of $\mathbb{R}^n$

Example 3.4

1. $\{e_1 = (1,0), e_2 = (0,1)\}$ *est une base de* $\mathbb{R}^2$.
2. $\{e_1 = (1,0,0), e_2 = (0,1,0), e_3 = (0,0,1)\}$ *is a basis of* $\mathbb{R}^3$.

Definition 3.9 *(Dimension of a Vector Space)*
The dimension of a vector space, denoted as $dimE$ is equal to the cardinality of its basis.

It is recalled that the cardinality of a set is the number of elements in that set.

Example 3.5 $\dim \mathbb{R}^2 = 2$, $dim\mathbb{R}^n = n$.

Remark 3.1 *By convention, we define* $dim\,\{0_E\} = 0$.

Remark 3.2 *Searching for a basis for a vector space E is to find a family of vectors in E in such a way that this family is both a linearly independent and generating family of E. The number of elements in this basis is the dimension of the space E.*

Theorem 3.10 *In a vector space of dimension n, a basis for E is a family that,*

1. *is free,*
2. *a generating family,*
3. *Contains n vectors*

and any family that satisfies two of the three previous properties is a basis for E.

Example 3.6 *Let $\beta = \{(1,1,1), (-1,1,1), (0,1,-1)\}$. The elements of β are vectors of $\mathbb{R}^3$ and β contains three vectors. According to the theorem 3.10, to prove that β is a basis for $\mathbb{R}^3$, it is enough to show that the family is linearly independent and generating, since $card(\beta) = dim\mathbb{R}^3$.*

Theorem 3.11 *Let E be a $\mathbb{K}-$ vector space of dimension n. If F is a vector subspace of E then $dimF \leq n$, and if in addition $dimF = n$ then $E = F$.*

Theorem 3.12 *Let E be a $\mathbb{K}-$ vector space of dimension n, and F_1, F_2 two v.s.s of E, then*

$$dim\,(F_1 + F_2) = dimF_1 + dimF_2 - dim\,(F_1 \cap F_2)$$

and

$$dim\,(F_1 \oplus F_2) = dimF_1 + dimF_2$$

4 Linear Applications

4.1 Definitions and Properties

Definition 4.1 *Let E, G two $\mathbb{K}-$vector spaces and f an application of E into G. We say that f is a linear application if and only if one of the two properties are satisfied*

1. $\forall x, y \in E, \forall \lambda, \mu \in \mathbb{K} \quad f(\lambda x + \mu y) = \lambda f(x) + \mu f(y)$.
2. $\begin{cases} \forall x, y \in E, & f(x+y) = f(x) + f(y) \\ \forall x \in E, \forall \lambda \in \mathbb{K}, & f(\lambda x) = \lambda f(x). \end{cases}$

Example 4.1

$$f:\ \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$
$$(x, y) \longmapsto f(x, y) = (x + y, x - y, 2x)$$

Prove that f is linear

Solution 4.2 *We prove that*

$$\begin{cases} \forall x, y \in E, & f(x+y) = f(x) + f(y) \\ \forall x \in E, \forall \lambda \in \mathbb{R} & f(\lambda x) = \lambda f(x). \end{cases}$$

Let $(x,y)\,,(x',y')\in\mathbb{R}^2$, *we have* $(x,y)+(x',y')=(x+x',y+y')$

$$\begin{aligned} f\left(x+x',y+y'\right) &= \Big(x+x'+y+y',x+x'-y-y',2\left(x+x'\right)\Big) \\ &= \big(x+y,x-y,2x\big)+\big(x'+y',x'-y',2x'\big) \\ &= f\left(x,y\right)+f\left(x',y'\right) \end{aligned}$$

Let $(x,y)\in\mathbb{R}^2$, *and* $\lambda\in\mathbb{R}$

$$\begin{aligned} f\big(\lambda\left(x,y\right)\big) &= f\left(\lambda x,\lambda y\right) \\ &= \Big(\lambda x+\lambda y,\lambda x-\lambda y,2\lambda x\Big) \\ &= \Big(\lambda\left(x+y\right),\lambda\left(x-y\right),\lambda\left(2x\right)\Big) \\ &= \lambda\Big(x+y,x-y,2x\Big) \\ &= \lambda f\left(x,y\right). \end{aligned}$$

So, f *is a linear application.*

Proposition 4.3 *Let* $f:\ E\longrightarrow G$ *is a linear application.*

1. $f(0_E)=0_G$.
2. $\forall x\in E,\quad f(-x)=-f(x)$.

Proof. 1) Since $0_E=0_E+0_E$

$$\begin{aligned} f(0_E) &= f\left(0_E+0_E\right) \\ &= f(0_E)+f(0_E) \\ &= 2f(0_E) \end{aligned}$$

$$\text{then}\quad f(0_E)=0_G$$

2) Let $x\in E$, we have

$$f\left(x-x\right)=f(0_E)=0_G\quad ....(1)$$

and since f est linear it yields

$$f\left(x-x\right)=f\left(x+(-x)\right)=f(x)+f(-x)\quad ....(2)$$

from (1) and (2) :

$$f(x)+f(-x)=0_G$$

and consequently

$$f(-x)=-f(x).$$

■

Space of Linear Applications

We denote by $\mathcal{L}(E,G)$ The set of all linear applications from$E$ into G. This set is equipped with an internal composition law denoted by $(+)$ and an external law $(.)$ defined as follows:
Soient $f,g \in \mathcal{L}(E,G)$ et $\lambda \in \mathbb{K}$.

$$\forall x \in E, \quad (f+g)(x) = f(x) + g(x) \quad \text{et} \qquad (\lambda.f)(x) = \lambda.f(x)$$

Proposition 4.4 *$(\mathcal{L}(E,G), +, .)$ est un $\mathbb{K}-$ vector space.*

4.2 Kernel and Image

Let $f: \ E \longrightarrow G$ be a linear application.

Definition 4.5 *(Kernel and iimage of a linear application)*

- *The kernel of f is denoted by ker, f and is defined as follows*

  $$ker\, f = \left\{ x \in E \mid f(x) = 0_G \right\}$$

  We also denote $kerf$ by $\ ker\, f = f^{-1}(0_G)$

- *The image of f is the set denoted by $im\, f$ and is defined as*

  $$im\, f = \left\{ y \in G \mid y = f(x) \quad où\; x \in E \right\} = f(E).$$

- *The rank of f is the dimension of $im\, f$, and it is written as $rg(f) = dim(im\, f)$.*

Example 4.2

$$f: \ \mathbb{R}^2 \longrightarrow \mathbb{R}$$
$$(x,y) \longmapsto x + 2y$$

Prove that f is linear and give its kernel and image.

Solution 4.6

$$\begin{aligned} kerf &= \left\{ (x,y) \in \mathbb{R}^2 \mid f(x) = 0_G \right\} \\ &= \left\{ (x,y) \in \mathbb{R}^2 \mid x + 2y = 0_G \right\} \\ &= \left\{ (x,y) \in \mathbb{R}^2 \mid x = -2y \right\} \end{aligned}$$

therefore

$$\begin{aligned} kerf &= \left\{ (-2y, y) \mid y \in \mathbb{R} \right\} \\ &= \left\{ y\,(-2,1) \mid y \in \mathbb{R} \right\}. \end{aligned}$$

We can write $\quad kerf = \langle(-2,1)\rangle$

$$im\, f = \left\{ u \in \mathbb{R} \mid u = f\,(x,y) \quad where\ (x,y) \in \mathbb{R}^2 \right\}$$

$$im\, f = \left\{ u \in \mathbb{R} \mid u = x + 2y \quad where\ (x,y) \in \mathbb{R}^2 \right\}$$

Properties

Let E, G be two $\mathbb{K}-$ vector spaces $f:\ E \longrightarrow G$ is a linear application, we have:

1. $\ker f$ is a vector subspace of E.
2. $im\, f$ is a vector subspace of G.

Proof. 1)
- Since f is a linear application, it yields $f(0_E) = 0_G$, then $0_E \in kerf$ and therefore $kerf \neq \emptyset$.
- Let $x_1, x_2 \in kerf$ then we have $f(x_1) = 0$ and $f(x_2) = 0$. Since f is linear, we have

$$f\,(x_1 + x_2) = f(x_1) + f(x_2) = 0 + 0 = 0.$$

$$\text{Donc } \ x_1 + x_2 \in kerf$$

- Let $x \in kerf, \lambda \in \mathbb{K},\ \ f(\lambda x) = \lambda f(x) = \lambda \times 0 = 0.$

$$\text{Then } \ \lambda x \in kerf.$$

$$kerf \ \text{ is a vector subspace of } E.$$

2)
- $0_G \in imf$ then $imf \neq \emptyset$.
- Let $y_1, y_2 \in imf$ then it exist x_1, x_2 in E such that $y_1 = f(x_1)$ and $y_2 = f(x_2)$.

$y_1 + y_2 = f(x_1) + f(x_2) = f\,(x_1 + x_2)$ and since $x_1 + x_2 \in E$, we deduce that $y_1 + y_2 \in imf$.
- Let $y \in imf, \lambda \in \mathbb{K}$, we have $\lambda y = \lambda f(x) = f(\lambda x) \in imf$ because $\lambda x \in E$.

$$imf \ \text{ is a vector subspace of } G.$$

■

Theorem 4.7 *(Injection- surjection)*

$$f \ \ \textit{is injective} \ \ \Longleftrightarrow \textit{ker}\ f = \{0_E\}$$

If dim $G = p$ *(finite), then*

$$f \ \ \textit{is surjective} \ \ \Longleftrightarrow \textit{dim im}\ f = \textit{dim}\ G = p.$$

In other words,

$$f \ \ \textit{is surjective} \ \ \Longleftrightarrow \textit{im}\ f = G.$$

Theorem 4.8 *(Fundamental theorem)*
Let $f: E \longrightarrow G$ *be a linear application such that* $\dim E = n$*(finite), then*

$$\dim E = \dim imf + \dim kerf$$

Example 4.3 *Let us consider the following application*

$$\begin{aligned} f: \ \ & \mathbb{R}^3 \longrightarrow \mathbb{R}^2 \\ & (x, y, z) \longrightarrow f(x, y, z) = (x + 2y, 2x + 3z) \end{aligned}$$

Determine $kerf, imf$ *and provide* $\dim kerf$ *et* $rg(f)$.

Solution 4.9

$$\begin{aligned} kerf &= \Big\{ (x, y, z) \in \mathbb{R}^3 \mid f(x, y, z) = 0_{\mathbb{R}^2} \Big\} \\ &= \Big\{ (x, y, z) \in \mathbb{R}^3 \mid \ (x + 2y, 2x + 3z) = (0, 0) \Big\} \\ &= \Big\{ (x, y, z) \in \mathbb{R}^3 \mid x + 2y = 0 \quad and \ \ 2x + 3z = 0 \Big\} \\ &= \Big\{ (x, y, z) \in \mathbb{R}^3 \mid y = -\frac{1}{2}x \quad et \ \ z = -\frac{2}{3}x \Big\} \\ &= \Big\{ \left(x, -\frac{1}{2}x, -\frac{2}{3}x \right) \ \mid \ x \in \mathbb{R} \Big\} \\ &= \Big\{ x \left(1, -\frac{1}{2}, -\frac{2}{3} \right) \ \mid \ x \in \mathbb{R} \Big\} \\ &= \langle \left(1, -\frac{1}{2}, -\frac{2}{3} \right) \rangle. \end{aligned}$$

$imf = \Big\{ f(x, y, z) \ \mid \ (x, y, z) \in \mathbb{R}^3 \Big\}$

$$\begin{aligned} f(x, y, z) &= (x + 2y, 2x + 3z) \ \mid \ (x, y, z) \in \mathbb{R}^3 \\ &= (x, 2x) + (2y, 0) + (0, 3z) \\ &= x(1, 2) + y(2, 0) + z(0, 3) \\ &= \langle (1, 2), (2, 0), (0, 3) \rangle \end{aligned}$$

we have

$$dim\,\mathbb{R}^3 = dim\,imf + dim\,kerf$$

$dim\,\mathbb{R}^3 = 3$ *and* $dim\,kerf = 1$ *then* $rg(f) = dim\,imf = 2.$