Khemis Miliana university
Faculty of Science and Technology
Department of Material Sciences

Level : L1

Specialization : ST +SM.

Module : Mathematics 1

Semestre1

Exercises Series N°2: Relations and Applications

I. Relations

Exercise n°1 : Let R be a relation defined on $\mathbb{R}$ by:

$$\forall x, y \in \mathbb{R}, \quad xRy \iff x(3+y^2) = y(3+x^2).$$

1. Prove that R is an equivalence relation on $\mathbb{R}$.
2. Determine $\overline{2}$ the equivalence class of the integer 2.

Exercise n°2 : Let R be a relation defined on $\mathbb{N}$ by:

$$\forall x, y \in \mathbb{N}, \quad xRy \iff \frac{2x+y}{3} \in \mathbb{N}.$$

1. Determine if $7R5$, $6R9$, $4R4$.
2. Prove that R is an equivalence relation on $\mathbb{N}$

Exercise n°3:

Let E and F be two sets and $f: E \longrightarrow F$ be a function

We define a relation R on E by: $\forall x, x' \in E, \quad xRx' \iff f(x) = f(x')$.

1. Prove that R is an equivalence relation on E
2. Describe the class $\bar{a}$ of the element $a \in E$
3. Describe the class $\bar{a}$ of the element $a \in E$ if the function f is injective.

Exercise n°4 :

Determine if the relations R below are order relations:

1. $\forall x, y \in \mathbb{R},\ \ xRy \Leftrightarrow e^x \leq e^y$.
2. $\forall x, y \in \mathbb{R},\ \ xRy \Leftrightarrow |x+1| \leq |y+1|$.
3. $\forall x, y \in]1, +\infty[,\ \ xRy \Leftrightarrow \frac{x}{1+x^2} \geq \frac{y}{1+y^2}$.
4. $\forall x, y \in \mathbb{R},\ \ xRy \Leftrightarrow x - y \in \mathbb{N}$.
5. $\forall x, y \in \mathbb{R},\ \ xRy \Leftrightarrow x - y \in \mathbb{Z}$.

II. Functions

Exercise n°5 :

1. Provide a counterexample to show that the following functions are not injective on $\mathbb{R}$

   $a)\ f(x) = \sin(2x) + 3$ $\qquad b)\ g(x) = |x^2 - 5x + 6|$ $\qquad c)\ h(x) = \dfrac{x^4}{4 + x^2}$

2. Solve the following equations in $\mathbb{R}$:
   $f(x) = 5,\quad g(x) = -7$ et $\quad h(x) = -1$.
   What can be deduced about the surjectivity of these functions?

Exercice n°6: Let's consider the function $f: \mathbb{R} - \{1/2\} \longrightarrow \mathbb{R}$ defined by:

$$f(x) = \frac{x+1}{2x-1}$$

1. Show that f is injective and determine if f is surjective.
2. Find the set F such that f is bijective from $\mathbb{R} - \{1/2\}$ to F, then calculate the inverse function f^{-1}
3. Determine the composed function $f \circ f$ and find by using a second method the inverse function $f^{-1}: F \longrightarrow \mathbb{R} - \left\{\frac{1}{2}\right\}$.

Exercice n°7: Let's consider the function $f: \mathbb{R} \longrightarrow \mathbb{R}$ defined by: $f(x) = \frac{x}{1+x^2}$.

1. Determine the direct image $f(A_1)$ et $f(A_2)$ where $A_1 = \{\ 0,\ \frac{1}{4},\ \sqrt{8},\ 4\}$
   and $A_2 = [2,\ 3]$.
2. Determine the inverse image $f^{-1}(B_1)$, $f^{-1}(B_2)$ with
   $B_1 = \{-1\},\quad B_2 = \{0, 1/2\}$.
3. Is the function f injective? Surjective? Justify
4. Prove that $f:]1, +\infty[\longrightarrow]0, 1/2[$ is bijective, and determine its inverse function f^{-1}