Series 4 : Algebraic Structures

Exercise 1 : 1. We equip $\mathbb{R}$ with the internal composition law $*$ defined by :

$$\forall x, y \in \mathbb{R}, \quad x * y = xy + (x^2-1)(y^2-1)$$

Show that $*$ is commutative, not associative, and that 1 is the identity element.

2. We equip $\mathbb{R}^+$ with the internal composition law $*$ defined by :

$$\forall x, y \in \mathbb{R}^+, \quad x * y = \sqrt{x^2+y^2}$$

Show that $*$ is commutative, associative, and that 0 is the identity element. Show that no element of $\mathbb{R}^+$ has an inverse for $*$.

3. We equip $\mathbb{R}$ with the internal composition law $*$ defined by :

$$\forall x, y \in \mathbb{R}, \quad x * y = \sqrt[3]{x^3+y^3}$$

Show that the map $x \mapsto x^3$ is an isomorphism from $(\mathbb{R}, *)$ to $(\mathbb{R}, +)$.
Deduce that $(\mathbb{R}, *)$ is a commutative group.

Exercise 2 : Let $G = \mathbb{R}^* \times \mathbb{R}$ and $*$ be the law in G defined by $(x, y) * (x', y') = (xx', xy' + y)$.

1. Show that $(G, *)$ is a non-commutative group. 2. Show that $((0, +\infty) \times \mathbb{R}, *)$ is a subgroup of $(G, *)$.

Exercise 3 : We equip $A = \mathbb{R} \times \mathbb{R}$ with two laws defined by :

$$(x, y) + (x', y') = (x + x', y + y') \quad \text{and} \quad (x, y) * (x', y') = (xx', xy' + x'y)$$

1. Show that $(A, +)$ is a commutative group.
2. a) Show that the law $*$ is commutative.
b) Show that $*$ is associative.
c) Determine the identity element of A for the law $*$.
d) Show that $(A, +, *)$ is a commutative ring.

Exercise 4 : Show that the intersection of two subgroups H and K of G is a subgroup of G.