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Introduction to Topology

Course Notes and Solved Exercises

Designed for Students of
Second year licence in mathematics

Presented by

Dr. Mihoub BOUDERBALA

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Preface

This document is intended as a clear and structured pedagogical resource for second-year undergraduate students enrolled in the *Introduction to Topology* course at the University of Djilali Bounaama in Khemis Miliana, Faculty of Mathematical Sciences and Computer Science.

Since the 2023–2024 academic year, I have had the privilege of teaching this course. The present manuscript is the culmination of my efforts to render the abstract ideas of general topology more accessible and engaging for learners encountering them for the first time.

The material is organized into four main chapters. Each chapter includes a selection of carefully chosen exercises—most accompanied by detailed solutions—to foster active learning and support independent study. In addition, representative past examination problems are provided to help students prepare effectively for final assessments.

The pedagogical approach emphasizes a gradual and intuitive development of core topological concepts. The first chapter introduces the foundational notions of open and closed sets, which serve as the building blocks of topological spaces, and proceeds systematically to subspaces and other derived constructions. This step-by-step progression is designed to help students develop both technical proficiency and geometric intuition.

This is a first version of the manuscript, and as such, it is open to refinement. I warmly welcome feedback, corrections, and suggestions from students, colleagues, and readers. Your insights will be invaluable in enhancing the clarity, accuracy, and utility of this text for future cohorts.

Finally, I would like to express my sincere gratitude to all those who have supported this endeavor, and especially to my students, whose curiosity and dedication continue to inspire my teaching. The Author

Introduction

Based on my experience teaching Topology to second-year undergraduate mathematics students, I have observed recurring challenges in grasping the abstract nature of the subject. This document is born from a desire to address these difficulties by providing a clear, structured, and accessible exposition of the fundamental concepts of general topology.

The primary aim of these notes is to serve as a reliable companion for students, helping them to build a solid conceptual foundation, navigate the common pitfalls of the subject, and ultimately strengthen their problem-solving abilities.

The content is organized into five chapters, each designed to progressively develop the theory while maintaining a balance between abstraction and intuition.

Chapter 1 lays the groundwork with the axiomatic definition of a topological space. It introduces essential notions such as interior, closure, boundary, and neighborhoods, all of which rely on a fluent understanding of elementary set theory. The chapter then establishes the concepts of limits and continuity, highlighting the crucial role of the Hausdorff (separated) property in ensuring the uniqueness of limits. A key result, that every open interval in \mathbb{R} is homeomorphic to \mathbb{R} itself, provides an early illustration of the power of topological equivalence. The chapter concludes with a treatment of the subspace topology and the product topology.

Chapter 2 focuses on metric spaces, demonstrating how a distance function naturally induces a topology. This setting provides a more concrete and intuitive framework for understanding limits, continuity, and convergence. The chapter explores metric-specific concepts such as uniform continuity, Lipschitz mappings, and isometries, and extends the notion of distance to sets via the diameter and distance between subsets. It also clarifies the relationship between metric and topological equivalence of metrics and establishes that every metric space is Hausdorff, a property that does not hold for all topological spaces.

Chapter 3 is devoted to compactness, a central and powerful topological property. While the general definition via open coverings (the Borel–Lebesgue property) is foundational, the chapter emphasizes more tangible characterizations in the context of metric spaces, notably the Bolzano–Weierstrass property (sequential compactness). The intrinsic nature of compactness is highlighted, along with its fundamental connection to continuity, as encapsulated in theorems on the compactness of products and the behavior of continuous functions on compact sets.

Chapter 4 explores the concept of connectedness, which captures the idea of a space being "in one piece." The chapter presents several equivalent definitions and examines related notions such as local connectedness and path-connectedness, supported by illustrative examples. A major result is the generalization of the Intermediate Value Theorem to continuous functions on connected spaces, demonstrating its topological underpinning.

Finally, Chapter 5 introduces normed vector spaces, which blend the geometric structure of a metric with the algebraic structure of a vector space. This leads naturally to the study of Banach spaces, a cornerstone of functional analysis where the "vectors" are often functions themselves.

These notes are designed to guide students from the foundational axioms of topology towards its profound applications, fostering both a technical mastery and a deeper conceptual understanding of the subject.

Chapter 1

Topological Spaces

In this chapter, we introduce the fundamental definition of a topological space and establish the essential vocabulary of general topology.

1.1 The Order Topology

Definition 1.1. Let (E, \leq) be a totally ordered set. The *open intervals* in E are subsets of one of the following four types:

1. E itself,
2. \emptyset ,
3. $]a, b[:= \{x \in E \mid a < x < b\}$ for $a, b \in E$,
4. $\{x \in E \mid x < b\}$ for some $b \in E$,
5. $\{x \in E \mid a < x\}$ for some $a \in E$.

The **order topology** on E is defined as the collection of all arbitrary unions of these open intervals.

Remark 1.2. If $E = \mathbb{R}$ with its standard order, the intervals of types (4) and (5) are conventionally denoted by $] - \infty, b[$ and $]a, +\infty[$, respectively.

If E is the extended real line $\mathbb{R} = \mathbb{R} \cup \{-\infty, +\infty\}$ (with the natural order $-\infty < x < +\infty$ for all $x \in \mathbb{R}$), then these intervals become $[-\infty, b[$ and $]a, +\infty]$, respectively.

1.2 Topological Spaces

Definition 1.3. A **topological space** is a pair (E, \mathcal{T}) where E is a set and $\mathcal{T} \subseteq \mathcal{P}(E)$ is a family of subsets of E , called *open sets*, satisfying the following axioms:

1. $\emptyset \in \mathcal{T}$ and $E \in \mathcal{T}$.
2. The intersection of any two open sets is open: if $U, V \in \mathcal{T}$, then $U \cap V \in \mathcal{T}$.
3. The union of any collection of open sets is open: if $\{U_i\}_{i \in I} \subseteq \mathcal{T}$, then $\bigcup_{i \in I} U_i \in \mathcal{T}$.

The family \mathcal{T} is called a *topology* on E .

1. The elements of \mathcal{T} are called the *open sets* of the topology. The pair (X, \mathcal{T}) is called a *topological space*.

2. The complements of open sets in E are called **closed sets**. Thus, a subset $F \subseteq E$ is closed if and only if $E \setminus F \in \mathcal{T}$.
3. Axiom (ii) implies by induction that any *finite* intersection of open sets is open. In other words, \mathcal{T} is closed under finite intersections.
4. Dually, the collection of closed sets is closed under arbitrary intersections and finite unions. However, an arbitrary intersection of open sets need not be open, and an arbitrary union of closed sets need not be closed. Classic counterexamples in \mathbb{R} with its standard topology \mathcal{T}_u are:

$$\bigcap_{n \in \mathbb{N}^*} \left] -\frac{1}{n}, \frac{1}{n} \right[= \{0\} \quad (\text{a closed set, not open}),$$

$$\bigcup_{n \in \mathbb{N}^*} \left[\frac{1}{n}, 1 \right[=]0, 1[\quad (\text{neither open nor closed}).$$

1. **The Standard Topology on \mathbb{R} :** A subset $U \subseteq \mathbb{R}$ is declared open if for every $x \in U$, there exists $\varepsilon > 0$ such that the open interval $]x - \varepsilon, x + \varepsilon[$ is contained in U . This defines a topology \mathcal{T}_u on \mathbb{R} , known as the *standard topology*.
2. The collection $\{\emptyset, \mathbb{R}\} \cup \{]a, b[\mid a, b \in \mathbb{R}, a < b\}$ is *not* a topology on \mathbb{R} because it is not closed under arbitrary unions (e.g., the union of all $]a, b[$ with $a < 0 < b$ is \mathbb{R} , which is included, but the union of all $]1/n, 1[$ for $n \in \mathbb{N}^*$ is $]0, 1[$, which *is* included; a better example is that the union of all $]x - 1, x + 1[$ for $x \in \mathbb{R} \setminus \mathbb{Q}$ is $\mathbb{R} \setminus \mathbb{Q}$, which is not an open interval). The main point is that the set of all open intervals alone is not a topology; one must include all possible unions of them.
3. On any set E , there are two canonical topologies:
 - The **discrete topology** $\mathcal{T}_d = \mathcal{P}(E)$, where every subset is open.
 - The **trivial topology** $\mathcal{T}_t = \{\emptyset, E\}$, where only the empty set and the whole space are open.
4. On a two-point set $E = \{a, b\}$, there are exactly four distinct topologies:

$$\begin{aligned} \mathcal{T}_t &= \{\emptyset, E\}, \\ \mathcal{T}_1 &= \{\emptyset, \{a\}, E\}, \\ \mathcal{T}_2 &= \{\emptyset, \{b\}, E\}, \\ \mathcal{T}_d &= \{\emptyset, \{a\}, \{b\}, E\}. \end{aligned}$$

5. Let (X, \mathcal{T}) be a topological space such that $\{x\} \in \mathcal{T}$ for every $x \in X$ (i.e., every singleton is open). Then (X, \mathcal{T}) is the discrete topology.

Exercise 1. Let E be an arbitrary set. Show that the family consisting of the empty set and the complements of finite subsets of E forms a topology on E .

Solution 1. Denote this family by $\mathcal{T} = \{E \setminus F \mid F \subseteq E \text{ is finite}\} \cup \{\emptyset\}$.

- Since \emptyset is finite, $E \setminus \emptyset = E \in \mathcal{T}$. Also, $\emptyset \in \mathcal{T}$ by definition.

- Let $\{U_i\}_{i \in I} \subseteq \mathcal{T}$ be an arbitrary family of open sets. For each i , either $U_i = \emptyset$ or $U_i = E \setminus F_i$ for some finite $F_i \subseteq E$. Their union is

$$\bigcup_{i \in I} U_i = E \setminus \left(\bigcap_{i \in I} F_i \right).$$

The intersection $\bigcap_{i \in I} F_i$ is a subset of any F_i , hence finite. Thus, the union is in \mathcal{T} .

- Let $U_1, \dots, U_n \in \mathcal{T}$ be a finite collection. For each k , write $U_k = E \setminus F_k$ with F_k finite (or $U_k = \emptyset = E \setminus E$, and we can take $F_k = E$ which is fine for the complement). Their intersection is

$$\bigcap_{k=1}^n U_k = E \setminus \left(\bigcup_{k=1}^n F_k \right).$$

The union $\bigcup_{k=1}^n F_k$ is a finite union of finite sets, hence finite. Thus, the intersection is in \mathcal{T} .

Therefore, \mathcal{T} is a topology on E , called the *cofinite topology*.

Exercise 2. Show that the set \mathcal{T} , consisting of \emptyset , \mathbb{R} , and arbitrary unions of intervals of the form $]a, b[$, is indeed a topology on \mathbb{R} .

Solution 2. This collection \mathcal{T} is precisely the standard topology on \mathbb{R} .

- $\emptyset \in \mathcal{T}$ (as the empty union) and $\mathbb{R} \in \mathcal{T}$ since $\mathbb{R} = \bigcup_{n=1}^{\infty}]-n, n[$.
- Let $U, V \in \mathcal{T}$. Then $U = \bigcup_{i \in I}]a_i, b_i[$ and $V = \bigcup_{j \in J}]c_j, d_j[$ for some index sets I, J . Using the distributive law,

$$U \cap V = \bigcup_{i \in I, j \in J} (]a_i, b_i[\cap]c_j, d_j[).$$

The intersection of two open intervals is either empty or another open interval $]e, f[$. Thus, $U \cap V$ is a union of open intervals, so $U \cap V \in \mathcal{T}$.

- Let $\{U_i\}_{i \in I} \subseteq \mathcal{T}$. Each U_i is a union of open intervals, so their total union $\bigcup_{i \in I} U_i$ is also a union of open intervals, hence in \mathcal{T} .

Therefore, \mathcal{T} is a topology on \mathbb{R} .

Exercise 3. What conditions must two distinct subsets A and B of a set E satisfy for $\mathcal{T} = \{\emptyset, A, B, E\}$ to be a topology on E ?

Solution 3. For \mathcal{T} to be a topology, it must be closed under finite intersections and arbitrary unions. The non-trivial checks are for $A \cap B$ and $A \cup B$.

Since \mathcal{T} has only four elements, $A \cap B$ and $A \cup B$ must each be one of \emptyset, A, B, E .

- $A \cap B \in \mathcal{T}$ implies $A \cap B = \emptyset, A$, or B .
- $A \cup B \in \mathcal{T}$ implies $A \cup B = A, B$, or E .

Since A and B are distinct, we cannot have both $A \subseteq B$ and $B \subseteq A$. The only consistent possibilities are:

$$A \subseteq B \quad \text{or} \quad B \subseteq A.$$

To see this, suppose $A \not\subseteq B$ and $B \not\subseteq A$. Then $A \cap B$ is a proper subset of both A and B , so $A \cap B \notin \mathcal{T}$. Also, $A \cup B$ is a proper superset of both A and B , so $A \cup B \notin \mathcal{T}$.

Conversely, if $A \subseteq B$, then $A \cap B = A \in \mathcal{T}$ and $A \cup B = B \in \mathcal{T}$. All other intersections and unions are trivially in \mathcal{T} .

Therefore, the necessary and sufficient condition is that one of the sets is contained in the other: $A \subseteq B$ or $B \subseteq A$.

Exercise 4. Let $E :=]0, +\infty[$, and for every $\alpha > 0$, let $O_\alpha :=]0, \alpha[$. Consider the family $\mathcal{T} = \{\emptyset\} \cup \{O_\alpha \mid \alpha > 0\} \cup \{E\}$.

1. Show that \mathcal{T} constitutes a topology on E .
2. Determine the closed sets of the topological space (E, \mathcal{T}) .

Solution 4. 1. We verify the axioms of a topology.

- $\emptyset \in \mathcal{T}$ and $E \in \mathcal{T}$ by definition.
- Let $O_\alpha, O_\beta \in \mathcal{T}$. Then $O_\alpha \cap O_\beta =]0, \min(\alpha, \beta)[= O_{\min(\alpha, \beta)} \in \mathcal{T}$.
- Let $\{O_{\alpha_i}\}_{i \in I} \subseteq \mathcal{T}$. Their union is

$$\bigcup_{i \in I} O_{\alpha_i} = \left] 0, \sup_{i \in I} \alpha_i \right[.$$

If $\sup_{i \in I} \alpha_i = +\infty$, the union is $E \in \mathcal{T}$. Otherwise, it is $O_{\sup \alpha_i} \in \mathcal{T}$.

Thus, \mathcal{T} is a topology on E .

2. A set is closed if and only if its complement in E is open. The open sets are \emptyset , E , and $]0, \alpha[$ for $\alpha > 0$. Their complements are:

$$\begin{aligned} E \setminus \emptyset &= E, \\ E \setminus E &= \emptyset, \\ E \setminus]0, \alpha[&= [\alpha, +\infty). \end{aligned}$$

Therefore, the closed sets are \emptyset , E , and all intervals of the form $[\alpha, +\infty)$ for $\alpha > 0$.

1.3 Neighborhood of a Point or a Set

Definition 1.4. Let (E, \mathcal{T}) be a topological space.

- A subset $V \subseteq E$ is a *neighborhood* of a point $a \in E$ if there exists an open set $O \in \mathcal{T}$ such that $a \in O \subseteq V$. The set of all neighborhoods of a is denoted by $\mathcal{V}(a)$.
- More generally, a *neighborhood* of a subset $A \subseteq E$ is any subset $V \subseteq E$ for which there exists an open set $O \in \mathcal{T}$ such that $A \subseteq O \subseteq V$. The set of all neighborhoods of A is denoted by $\mathcal{V}(A)$.

In mathematical notation, for any $a \in E$ and $A \subseteq E$,

$$V \in \mathcal{V}(a) \iff \exists O \in \mathcal{T} \text{ such that } a \in O \subseteq V,$$

$$V \in \mathcal{V}(A) \iff \exists O \in \mathcal{T} \text{ such that } A \subseteq O \subseteq V.$$

Example 1.5. In the topological space \mathbb{R} with its standard topology, a neighborhood of a point $a \in \mathbb{R}$ is any subset of \mathbb{R} that contains an open interval of the form $]a - \varepsilon, a + \varepsilon[$ for some $\varepsilon > 0$.

Example 1.6. Consider the set $E = \{a, b, c\}$ equipped with the topology $\mathcal{T} = \{\emptyset, \{a\}, E\}$. The neighborhoods of the point a are all subsets $V \subseteq E$ that contain the open set $\{a\}$. Thus, $\mathcal{V}(a) = \{\{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}\}$.

Remark 1.7. Let $a \in E$. The family of neighborhoods $\mathcal{V}(a)$ has the following properties:

1. Every neighborhood of a contains a .
2. Every superset of a neighborhood of a is also a neighborhood of a .
3. Any finite intersection of neighborhoods of a is a neighborhood of a .
4. Every open set containing a is a neighborhood of a .

The next theorem provides a fundamental characterization of open sets in terms of neighborhoods.

Theorem 1.8. A subset U of a topological space (E, \mathcal{T}) is open if and only if U is a neighborhood of each of its points.

Proof. (\Rightarrow) If U is open, then for every $a \in U$, we have $a \in U \subseteq U$, so U is a neighborhood of a .

(\Leftarrow) Conversely, suppose U is a neighborhood of each of its points. Then for every $a \in U$, there exists an open set $O_a \in \mathcal{T}$ such that $a \in O_a \subseteq U$. Consider the union $O = \bigcup_{a \in U} O_a$. This union O is open as it is a union of open sets. Moreover, for every $a \in U$, $a \in O_a \subseteq O$, so $U \subseteq O$. Conversely, since each $O_a \subseteq U$, we have $O \subseteq U$. Therefore, $U = O$, which is open. ■

Exercise 5. Show that in $\mathbb{N} \cup \{+\infty\}$ equipped with the order topology, the neighborhoods of $+\infty$ are precisely the subsets that contain $+\infty$ and all integers from some fixed rank onward.

Solution 5. In the order topology on $\mathbb{N} \cup \{+\infty\}$, a base of open neighborhoods for the point $+\infty$ is given by the intervals of the form $(a, +\infty] = \{x \in \mathbb{N} \cup \{+\infty\} \mid x > a\}$ for $a \in \mathbb{N}$.

Let $V \subseteq \mathbb{N} \cup \{+\infty\}$. By definition, V is a neighborhood of $+\infty$ if and only if there exists an open set O such that $+\infty \in O \subseteq V$. Since the intervals $(a, +\infty]$ form a neighborhood base, this is equivalent to the existence of some $a \in \mathbb{N}$ such that $(a, +\infty] \subseteq V$.

The set $(a, +\infty]$ contains $+\infty$ and all integers $n \in \mathbb{N}$ with $n > a$. Therefore, V is a neighborhood of $+\infty$ if and only if there exists $N \in \mathbb{N}$ (take $N = a + 1$) such that V contains $+\infty$ and all integers $n \geq N$.

Conversely, if a set V contains $+\infty$ and all integers $n \geq N$ for some N , then it contains the open interval $(N - 1, +\infty]$, so V is a neighborhood of $+\infty$.

Exercise 6. (Neighborhood Axioms) Let X be a set. Suppose that for every element $x \in X$, there is a family $\mathcal{V}(x)$ of subsets of X satisfying the following properties:

1. $\mathcal{V}(x) \neq \emptyset$ and for every $V \in \mathcal{V}(x)$, we have $x \in V$.
2. If a subset $A \subseteq X$ contains some $V \in \mathcal{V}(x)$, then $A \in \mathcal{V}(x)$.
3. The intersection of any two elements of $\mathcal{V}(x)$ is also in $\mathcal{V}(x)$.
4. For every $V \in \mathcal{V}(x)$, there exists $W \in \mathcal{V}(x)$ such that for all $y \in W$, we have $W \in \mathcal{V}(y)$.

Show that there exists a unique topology \mathcal{T} on X such that for every $x \in X$, $\mathcal{V}(x)$ is precisely the family of all neighborhoods of x in the topological space (X, \mathcal{T}) .

Solution 6. We will construct the topology \mathcal{T} from the given neighborhood systems and prove it is the unique one with the desired property.

Step 1: Define the candidate topology. Define a subset $U \subseteq X$ to be *open* if and only if it is a neighborhood of each of its points, i.e.,

$$U \in \mathcal{T} \quad \text{if and only if} \quad \forall x \in U, U \in \mathcal{V}(x).$$

Step 2: Show that \mathcal{T} is a topology. We verify the three axioms of a topology.

- $\emptyset, X \in \mathcal{T}$: The statement " $\forall x \in \emptyset, \emptyset \in \mathcal{V}(x)$ " is vacuously true, so $\emptyset \in \mathcal{T}$.
For X , let $x \in X$ be arbitrary. By (V1), there exists some $V \in \mathcal{V}(x)$. By (V2), since $V \subseteq X$, we have $X \in \mathcal{V}(x)$. As this holds for all $x \in X$, $X \in \mathcal{T}$.
- \mathcal{T} is closed under arbitrary unions: Let $\{U_i\}_{i \in I} \subseteq \mathcal{T}$ and let $U = \bigcup_{i \in I} U_i$. Let $x \in U$. Then $x \in U_j$ for some $j \in I$. Since $U_j \in \mathcal{T}$, we have $U_j \in \mathcal{V}(x)$. But $U_j \subseteq U$, so by (V2), $U \in \mathcal{V}(x)$. Since this holds for all $x \in U$, $U \in \mathcal{T}$.
- \mathcal{T} is closed under finite intersections: Let $U_1, U_2 \in \mathcal{T}$ and let $U = U_1 \cap U_2$. Let $x \in U$. Then $x \in U_1$ and $x \in U_2$, so $U_1 \in \mathcal{V}(x)$ and $U_2 \in \mathcal{V}(x)$. By (V3), $U = U_1 \cap U_2 \in \mathcal{V}(x)$. Since this holds for all $x \in U$, $U \in \mathcal{T}$. The result extends to any finite intersection by induction.

Thus, \mathcal{T} is a topology on X .

Step 3: Show that $\mathcal{V}(x)$ is the family of \mathcal{T} -neighborhoods of x . Let $x \in X$. We must show that a set V is a \mathcal{T} -neighborhood of x if and only if $V \in \mathcal{V}(x)$.

(\Leftarrow) Suppose $V \in \mathcal{V}(x)$. By (V4), there exists $W \in \mathcal{V}(x)$ such that for all $y \in W$, $W \in \mathcal{V}(y)$. This means, by our definition of \mathcal{T} , that W is an open set (i.e., $W \in \mathcal{T}$). Since $x \in W \subseteq V$, V contains an open set containing x , so V is a \mathcal{T} -neighborhood of x .

(\Rightarrow) Conversely, suppose V is a \mathcal{T} -neighborhood of x . Then there exists an open set $U \in \mathcal{T}$ such that $x \in U \subseteq V$. Since U is open and $x \in U$, we have $U \in \mathcal{V}(x)$ by the definition of \mathcal{T} . Then, by (V2), since $U \subseteq V$, we get $V \in \mathcal{V}(x)$.

Therefore, the two notions of neighborhood coincide.

Step 4: Uniqueness. Suppose \mathcal{T}' is another topology on X for which $\mathcal{V}(x)$ is the family of neighborhoods of x . A set U is open in \mathcal{T}' if and only if it is a neighborhood of each of its points (by Theorem 1.8), which is if and only if $U \in \mathcal{V}(x)$ for all $x \in U$. This is exactly the definition of our topology \mathcal{T} . Hence $\mathcal{T}' = \mathcal{T}$, proving uniqueness.

1.4 Bases of Open Sets and Neighborhood Bases

Definition 1.9. Let (E, \mathcal{T}) be a topological space, and let \mathcal{B} be a collection of subsets of E . We say that \mathcal{B} is a *basis* for the topology \mathcal{T} (or a *basis of open sets*) if one (and hence all) of the following equivalent conditions is satisfied:

1. The open sets in \mathcal{T} are precisely the unions of elements of \mathcal{B} .
2. $\mathcal{B} \subseteq \mathcal{T}$ and every open set in \mathcal{T} is a union of elements of \mathcal{B} .
3. $\mathcal{B} \subseteq \mathcal{T}$ and for every $U \in \mathcal{T}$ and every point $x \in U$, there exists a set $B \in \mathcal{B}$ such that $x \in B \subseteq U$.

Remark 1.10. The topology \mathcal{T} itself is always a basis for \mathcal{T} . However, it is usually more useful to work with a smaller, more economical basis. For example, the collection of all open intervals $]a, b[$ forms a basis for the standard topology on \mathbb{R} .

Example 1.11. The family

$$\left\{ \left[r - \frac{1}{n}, r + \frac{1}{n} \right] \mid r \in \mathbb{Q}, n \in \mathbb{N}^* \right\}$$

forms a countable basis for the standard topology on \mathbb{R} . This follows from the density of \mathbb{Q} in \mathbb{R} and the Archimedean property. A topological space that admits a countable basis is said to be *second-countable*.

Definition 1.12 (Fundamental System of Neighborhoods). A *fundamental system of neighborhoods* (or a *neighborhood basis*) of a point $a \in E$ is a family $\mathcal{B}(a)$ of neighborhoods of a such that:

$$\forall V \in \mathcal{V}(a), \exists W \in \mathcal{B}(a) \text{ such that } W \subseteq V.$$

Example 1.13. For every $a \in \mathbb{R}$, the family of intervals $\left\{ \left[a - \frac{1}{n}, a + \frac{1}{n} \right] \mid n \in \mathbb{N}^* \right\}$ is a countable neighborhood basis for a in the standard topology. This example shows that the elements of a neighborhood basis need not be open sets themselves.

It is often convenient to work with bases consisting of open sets. The following proposition establishes the precise link between a basis of open sets and neighborhood bases.

Proposition 1.14. Let (E, \mathcal{T}) be a topological space and let $\mathcal{B} \subseteq \mathcal{P}(E)$. Then \mathcal{B} is a basis of open sets for \mathcal{T} if and only if, for every point $a \in E$, the family

$$\mathcal{B}(a) := \{B \in \mathcal{B} \mid a \in B\}$$

is a neighborhood basis for a .

Proof. (\Rightarrow) Suppose \mathcal{B} is a basis of open sets. Let $a \in E$ and let V be a neighborhood of a . Then there exists an open set $U \in \mathcal{T}$ such that $a \in U \subseteq V$. Since \mathcal{B} is a basis, U is a union of elements of \mathcal{B} , so there exists $B \in \mathcal{B}$ with $a \in B \subseteq U \subseteq V$. Thus, $\mathcal{B}(a)$ is a neighborhood basis for a .

(\Leftarrow) Conversely, suppose that for every $a \in E$, $\mathcal{B}(a)$ is a neighborhood basis for a . Let $U \subseteq E$. By Theorem 1.8, U is open if and only if it is a neighborhood of each of its points. This means that for every $a \in U$, there exists $B_a \in \mathcal{B}(a) \subseteq \mathcal{B}$ such that $B_a \subseteq U$. Therefore, $U = \bigcup_{a \in U} B_a$, which is a union of elements of \mathcal{B} . Hence, \mathcal{B} is a basis of open sets. ■

Exercise 7. Show that $\mathcal{B} = \{[a, b[\mid (a, b) \in \mathbb{R}^2, a < b\} \cup \{\emptyset\}$ is a basis for a topology on \mathbb{R} .

Solution 7. To verify that \mathcal{B} is a basis, we must check two conditions.

(1) **Coverage of \mathbb{R} :** For any $x \in \mathbb{R}$, the interval $[x, x + 1[$ is in \mathcal{B} and contains x . Thus, $\bigcup_{B \in \mathcal{B}} B = \mathbb{R}$.

(2) **Local intersection property:** Let $B_1 = [a_1, b_1[$ and $B_2 = [a_2, b_2[$ be two elements of \mathcal{B} , and let $x \in B_1 \cap B_2$. Then $x \geq \max(a_1, a_2)$ and $x < \min(b_1, b_2)$. Define $B = [x, \min(b_1, b_2)[$. Then $B \in \mathcal{B}$, and we have $x \in B \subseteq B_1 \cap B_2$.

Since both conditions are satisfied, \mathcal{B} is a basis for a topology on \mathbb{R} . This topology is known as the *lower limit topology* or the *Sorgenfrey line*.

1.5 Interior and Closure

1.5.1 Closure

Definition 1.15. In a topological space E , a point x is said to be *adherent* to a subset $A \subseteq E$ if every neighborhood of x intersects A , i.e.,

$$\forall V \in \mathcal{V}(x), \quad V \cap A \neq \emptyset.$$

The set of all points adherent to A is called the *closure* of A and is denoted by \overline{A} .

Remark 1.16. Every element $x \in A$ is adherent to A (since $x \in V$ for any neighborhood V of x), so $A \subseteq \overline{A}$.

We can distinguish two types of adherent points.

Definition 1.17. An adherent point x of A is:

- an *isolated point* of A if there exists a neighborhood V of x such that $A \cap V = \{x\}$ (equivalently, $x \in A$ and $\{x\}$ is open in the subspace topology on A).
 - a *limit point* (or *accumulation point*) of A if every neighborhood of x intersects $A \setminus \{x\}$ (note that x need not belong to A).
1. A point cannot be both isolated and a limit point of A . If an adherent point x is not isolated, then for every neighborhood V of x , $A \cap V \neq \{x\}$, so x is a limit point. Thus, these are the only two possibilities for adherent points.
 2. Every limit point of A is adherent to A , but the converse is generally false (e.g., isolated points are adherent but not limit points).
 3. The set of all limit points of A is called the *derived set* of A , denoted by A' . This concept was introduced by the German mathematician Georg Cantor (1845–1918).

Example 1.18. 1. In \mathbb{R} with the standard topology, the closure of $]0, \sqrt{2}[\cup \{3\}$ is $[0, \sqrt{2}] \cup \{3\}$. The point 3 is an isolated point, while $[0, \sqrt{2}]$ is the set of limit points.

2. The closure of $\{1/n \mid n \in \mathbb{N}^*\}$ is $\{1/n \mid n \in \mathbb{N}^*\} \cup \{0\}$. The points $1/n$ are isolated, and 0 is a limit point.

Example 1.19. If $(E, \mathcal{T}) = (\mathbb{R}, \text{standard topology})$ and $A =]a, b[$, then $\overline{A} = [a, b]$. Indeed, $[a, b]$ is a closed set containing A , so $\overline{A} \subseteq [a, b]$. The only closed set among the possibilities $]a, b[$, $]a, b]$, $[a, b[$, and $[a, b]$ is $[a, b]$, hence $\overline{A} = [a, b]$.

Proposition 1.20. The closure of a subset A is the smallest closed set containing A . In particular, A is closed if and only if $A = \overline{A}$.

Proof. The family of closed sets containing A is nonempty (since E is such a set), so their intersection exists and is closed. We must show that $x \in \overline{A}$ if and only if x belongs to every closed set containing A .

A point x is adherent to A if every open neighborhood of x intersects A . This is equivalent to saying that there is no open set containing x that is disjoint from A , or, by taking complements, that there is no closed set containing A that excludes x . The contrapositive of this is that every closed set containing A must contain x . ■

Exercise 8. Let O be an open subset of a topological space (E, \mathcal{T}) . Show that for any subset $A \subseteq E$,

$$A \cap O = \emptyset \iff \overline{A} \cap O = \emptyset.$$

Solution 8. We prove both implications.

(\Rightarrow) Assume $A \cap O = \emptyset$. Since O is open, its complement $E \setminus O$ is closed. The hypothesis $A \cap O = \emptyset$ implies $A \subseteq E \setminus O$. Because the closure \overline{A} is the smallest closed set containing A , we have $\overline{A} \subseteq E \setminus O$. This is equivalent to $\overline{A} \cap O = \emptyset$.

(\Leftarrow) Assume $\overline{A} \cap O = \emptyset$. Since $A \subseteq \overline{A}$, it follows immediately that $A \cap O \subseteq \overline{A} \cap O = \emptyset$. Hence, $A \cap O = \emptyset$.

This completes the proof.

1.5.2 Interior

Definition 1.21. For a subset B of a topological space E , the *interior* of B , denoted by $\overset{\circ}{B}$, is the largest open set contained in B . Dually to the inclusion $A \subseteq \overline{A}$, we have the immediate inclusion $\overset{\circ}{B} \subseteq B$.

Corollary 1.22. 1. The interior of B is the set of points for which B is a neighborhood.

2. The interior of B is the union of all open sets contained in B (this union exists since \emptyset is open, though it may be empty). That is,

$$\overset{\circ}{B} = \bigcup_{\substack{O \in \mathcal{T} \\ O \subseteq B}} O.$$

3. A subset B of E is open if and only if $\overset{\circ}{B} = B$.

Theorem 1.23. A point $b \in E$ is in the interior of B (i.e., $b \in \overset{\circ}{B}$) if and only if B is a neighborhood of b .

Proof. (\Rightarrow) If $b \in \overset{\circ}{B}$, then $\overset{\circ}{B}$ is an open set with $b \in \overset{\circ}{B} \subseteq B$, so B is a neighborhood of b .

(\Leftarrow) If B is a neighborhood of b , there exists an open set O such that $b \in O \subseteq B$. By definition of the interior, $O \subseteq \overset{\circ}{B}$, so $b \in \overset{\circ}{B}$. ■

Proposition 1.24. Let E be a topological space and let A and B be subsets of E . Then:

1. If A and B are complements, then $\overset{\circ}{B}$ and \overline{A} are complements.
2. If $A \subseteq B$, then $\overline{A} \subseteq \overline{B}$ and $\overset{\circ}{A} \subseteq \overset{\circ}{B}$.
3. $\overline{A \cup B} = \overline{A} \cup \overline{B}$ and $A \overset{\circ}{\cap} B = \overset{\circ}{A} \cap \overset{\circ}{B}$.

Proof. 1. By Theorem 1.23, $x \notin \overset{\circ}{B} \iff B$ is not a neighborhood of $x \iff$ every neighborhood of x meets $A \iff x \in \overline{A}$.

2. Follows directly from the definitions of closure and interior.

3. The inclusion $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$ follows from (ii). The reverse inclusion holds because $A \subseteq A \cup B$ and $B \subseteq A \cup B$ imply $\overline{A} \subseteq \overline{A \cup B}$ and $\overline{B} \subseteq \overline{A \cup B}$. The proof for the interior of the intersection is analogous. ■

Remark 1.25. In general, we only have the inclusions $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$ and $A \overset{\circ}{\cup} B \supseteq \overset{\circ}{A} \cup \overset{\circ}{B}$, and equality does not always hold.

Example 1.26. If $A = [0, 1]$ and $B = [1, 2]$, then $A \overset{\circ}{\cup} B = [0, 2] \setminus \{1\}$. However, $\overset{\circ}{A} \cup \overset{\circ}{B} =]0, 1[\cup]1, 2[=]0, 2[\setminus \{1\}$. Conversely, if $A =]0, 1[$ and $B =]1, 2[$, then $A \cap B = \emptyset$, so $\overline{A \cap B} = \emptyset$. But $\overline{A} \cap \overline{B} = [0, 1] \cap [1, 2] = \{1\}$.

Definition 1.27. The *boundary* of a subset A of a topological space E is defined by

$$\text{Fr}(A) := \overline{A} \setminus \overset{\circ}{A} = \overline{A} \cap \overline{E \setminus A}.$$

It is equal to the boundary of the complement of A .

Example 1.28. If $E = \mathbb{R}$ with the standard topology and $A = [0, 1[$, then $\overline{A} = [0, 1]$, $\overset{\circ}{A} =]0, 1[$, and $\text{Fr}(A) = \{0, 1\}$.

Exercise 9. Let E be a topological space and let $A, B \subseteq E$.

1. Show that $\overline{A^c} = (\overset{\circ}{A})^c$ and that $\overset{\circ}{A^c} = (\overline{A})^c$.
2. Show that

$$\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}, \quad \overline{A \cup B} = \overline{A} \cup \overline{B}, \quad A \overset{\circ}{\cap} B = \overset{\circ}{A} \cap \overset{\circ}{B}.$$
3. What can be said about $\overset{\circ}{A} \cup \overset{\circ}{B}$?
4. Define $u(A) = \overset{\circ}{\overline{A}}$ and $v(A) = \overline{\overset{\circ}{A}}$.
 - (a) Compute $u(A)$ and $v(A)$ for $E = \mathbb{R}$ (with the standard topology) and for $A =]0, 2]$ and $A = \mathbb{Q}$.
 - (b) Compare the sets $A, \overset{\circ}{A}, u(A)$, and $v(A)$.

Solution 9. 1. The interior $\overset{\circ}{A}$ is the union of all open sets contained in A . Taking complements, $(\overset{\circ}{A})^c$ is the intersection of all closed sets containing A^c . By Proposition 1.20, this intersection is precisely $\overline{A^c}$. The second identity follows by replacing A with A^c in the first.

2. The inclusion $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$ is immediate from the monotonicity of closure (since $A \cap B \subseteq A$ and $A \cap B \subseteq B$).

For the union, we have $A, B \subseteq A \cup B$, so $\overline{A}, \overline{B} \subseteq \overline{A \cup B}$, hence $\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$. Conversely, $A \cup B \subseteq \overline{A} \cup \overline{B}$, and since the right-hand side is closed (as a finite union of closed sets), it must contain $\overline{A \cup B}$.

The identity for the interior of an intersection follows by applying the first part to the complements of A and B and using the duality between interior and closure.

3. The set $\overset{\circ}{A} \cup \overset{\circ}{B}$ is an open set contained in $A \cup B$, so $\overset{\circ}{A} \cup \overset{\circ}{B} \subseteq \overset{\circ}{A \cup B}$. The reverse inclusion is generally false. A standard counterexample is $A =]0, 1]$, $B =]1, 2]$ in \mathbb{R} . Then $\overset{\circ}{A \cup B} =]0, 2[$, but $\overset{\circ}{A} \cup \overset{\circ}{B} =]0, 1[\cup]1, 2[$.

4. [label=(d), leftmargin=*]

For $A =]0, 2]$:

$$\overline{A} = [0, 2], \quad u(A) = \overset{\circ}{\overline{A}} =]0, 2[.$$

$$\overset{\circ}{A} =]0, 2[, \quad v(A) = \overline{\overset{\circ}{A}} = [0, 2].$$

For $A = \mathbb{Q}$:

$$\overline{A} = \mathbb{R}, \quad u(A) = \overset{\circ}{\overline{A}} = \overset{\circ}{\mathbb{R}} = \mathbb{R}.$$

$$\overset{\circ}{A} = \emptyset, \quad v(A) = \overline{\overset{\circ}{A}} = \overline{\emptyset} = \emptyset.$$

- (b) In general, we have $\overset{\circ}{A} \subseteq v(A) \subseteq \overline{A}$ and $\overset{\circ}{A} \subseteq u(A) \subseteq \overline{A}$. However, $u(A)$ and $v(A)$ are not comparable in general. For $A = \mathbb{Q}$, $v(A) = \emptyset \subsetneq \mathbb{R} = u(A)$. For a closed ball in \mathbb{R}^n , $u(A) \subsetneq v(A)$.

Exercise 10. Consider \mathbb{R}^2 with its standard topology. Determine the interior and the closure of the following subsets:

$$A = \{(x, y) \in \mathbb{R}^2 \mid 2x > y + 1\},$$

$$B = \{(x, y) \in \mathbb{R}^2 \mid 0 < x < 2 \text{ and } 0 \leq y \leq 1\},$$

$$C = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x^2 + y^2 \leq 1\},$$

$$D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \geq 4\} \setminus \mathbb{Q}^2.$$

Solution 10. • The set A is open because it is the preimage of the open set $]1, \infty[$ under the continuous map $(x, y) \mapsto 2x - y - 1$. Hence, $\overset{\circ}{A} = A$. Its closure is $\overline{A} = \{(x, y) \mid 2x \geq y + 1\}$, as every point on the boundary line can be approached by a sequence from A .

- The set B is a rectangle missing its left and right edges but including its top and bottom edges. Its closure is the closed rectangle $\overline{B} = [0, 2] \times [0, 1]$. Its interior is the open rectangle $\overset{\circ}{B} =]0, 2[\times]0, 1[$.
- The set C is the closed unit disk, hence it is closed: $\overline{C} = C$. Its interior is the open unit disk $\overset{\circ}{C} = \{(x, y) \mid x^2 + y^2 < 1\}$. Note that the origin is an interior point.
- The set D is the exterior of the closed disk of radius 2, with all points having rational coordinates removed. Since \mathbb{Q}^2 is dense in \mathbb{R}^2 , removing it does not change the closure, so $\overline{D} = \{(x, y) \mid x^2 + y^2 \geq 4\}$. However, any nonempty open set in \mathbb{R}^2 contains points of \mathbb{Q}^2 , so no nonempty open set is contained in D . Thus, $\overset{\circ}{D} = \emptyset$.

Exercise 11. Let (E, \mathcal{T}) be a topological space and let $A, B \subseteq E$ such that $E = A \cup B$. Show that

$$E = \overset{\circ}{A} \cup \overline{B}.$$

Solution 11. We will show that every point $x \in E$ belongs to either $\overset{\circ}{A}$ or \overline{B} .

Let $x \in E$ be arbitrary. We consider two cases.

Case 1: $x \notin \overline{B}$.

If x is not in the closure of B , then by definition there exists an open neighborhood U of x such that $U \cap B = \emptyset$. This implies $U \subseteq B^c$. Since $E = A \cup B$, we have $B^c \subseteq A$, and therefore $U \subseteq A$. This means that x is an interior point of A , so $x \in \overset{\circ}{A}$.

Case 2: $x \in \overline{B}$.

In this case, the conclusion is immediate.

Since every point $x \in E$ is in either $\overset{\circ}{A}$ or \overline{B} , we conclude that $E = \overset{\circ}{A} \cup \overline{B}$.

Exercise 12. Determine the boundary of the following subsets of \mathbb{R}^2 :

$$A_1 = \{(x, y) \in \mathbb{R}^2 \mid 0 < x^2 + y^2 < 2\},$$

$$A_2 = \mathbb{Q} \times \mathbb{Q},$$

$$A_3 =]-2, 1[\times]0, 1[.$$

Solution 12. Recall that $\text{Fr}(S) = \overline{S} \setminus \overset{\circ}{S}$.

- $\overline{A_1} = \{(x, y) \mid x^2 + y^2 \leq 2\}$, $\overset{\circ}{A_1} = A_1$. Thus, $\text{Fr}(A_1) = \{(0, 0)\} \cup \{(x, y) \mid x^2 + y^2 = 2\}$.
- $\overline{A_2} = \mathbb{R}^2$, $\overset{\circ}{A_2} = \emptyset$. Thus, $\text{Fr}(A_2) = \mathbb{R}^2$.
- $\overline{A_3} = [-2, 1] \times [0, 1]$, $\overset{\circ}{A_3} =]-2, 1[\times]0, 1[$. Thus, $\text{Fr}(A_3)$ is the boundary of the rectangle, i.e.,

$$\text{Fr}(A_3) = \{-2, 1\} \times [0, 1] \cup [-2, 1] \times \{0, 1\}.$$

Exercise 13. Let $\mathcal{T} = \{]-\alpha, \alpha[\mid \alpha \in [0, +\infty)\}$, where we adopt the conventions $]-\infty, \infty[= \mathbb{R}$ and $]-0, 0[= \emptyset$.

1. Show that \mathcal{T} is a topology on \mathbb{R} .
2. Determine the closure, interior, and boundary of a singleton $\{a\}$ and of a closed interval $[a, b]$ (with $a \leq b$).

Solution 13. 1. We verify the three axioms of a topology.

- $\emptyset =] - 0, 0[\in \mathcal{T}$ and $\mathbb{R} =] - \infty, \infty[\in \mathcal{T}$.
- Let $U =] - \alpha, \alpha[$ and $V =] - \beta, \beta[$ be two elements of \mathcal{T} . Their intersection is

$$U \cap V =] - \min(\alpha, \beta), \min(\alpha, \beta)[\in \mathcal{T}.$$

By induction, \mathcal{T} is closed under finite intersections.

- Let $\{] - \alpha_i, \alpha_i[\}_{i \in I} \subseteq \mathcal{T}$ be an arbitrary family. Their union is

$$\bigcup_{i \in I}] - \alpha_i, \alpha_i[=] - \sup_{i \in I} \alpha_i, \sup_{i \in I} \alpha_i[\in \mathcal{T}.$$

Therefore, \mathcal{T} is a topology on \mathbb{R} . This is known as the *symmetric interval topology*.

2. First, we describe the open and closed sets in this topology.

- **Open sets:** By definition, the non-empty proper open sets are symmetric intervals $] - \alpha, \alpha[$ with $\alpha \in (0, \infty)$.
- **Closed sets:** The complements of open sets are the closed sets. They are \mathbb{R} , \emptyset , and the sets of the form $] - \infty, -\alpha] \cup [\alpha, \infty[$ for $\alpha \in [0, \infty)$.

Now we analyze the requested sets.

The singleton $\{a\}$ (with $a \neq 0$):

- *Interior:* The only open set contained in $\{a\}$ is \emptyset , so $\mathring{\{a\}} = \emptyset$.
- *Closure:* The closure $\overline{\{a\}}$ is the smallest closed set containing a . A closed set $] - \infty, -\alpha] \cup [\alpha, \infty[$ contains a if and only if $|a| \geq \alpha$. The smallest such closed set (for the inclusion order) is obtained when $\alpha = |a|$, so

$$\overline{\{a\}} =] - \infty, -|a|] \cup [|a|, \infty[.$$

- *Boundary:* The boundary is $\text{Fr}(\{a\}) = \overline{\{a\}} \setminus \mathring{\{a\}} = \overline{\{a\}}$.

The singleton $\{0\}$:

- *Interior:* $\mathring{\{0\}} = \emptyset$, for the same reason as above.
- *Closure:* Every non-empty open set contains 0, so 0 is in the closure of every non-empty set. The smallest closed set containing 0 is the whole space \mathbb{R} (since all other closed sets are of the form $] - \infty, -\alpha] \cup [\alpha, \infty[$ which do not contain 0). Thus, $\overline{\{0\}} = \mathbb{R}$.
- *Boundary:* $\text{Fr}(\{0\}) = \mathbb{R}$.

The closed interval $[a, b]$ (with $a \leq b$): We consider cases based on whether the interval contains 0.

Case 1: $0 \in [a, b]$ (i.e., $a \leq 0 \leq b$).

- *Interior:* The interior is the largest open set contained in $[a, b]$. The open sets are symmetric intervals around 0, so the largest one contained in $[a, b]$ is $] - \alpha, \alpha[$ where $\alpha = \min(|a|, b)$. Thus,

$$\mathring{[a, b]} =] - \min(|a|, b), \min(|a|, b)[.$$

- *Closure:* Since $[a, b]$ contains 0, its closure must be the entire space \mathbb{R} . This is because every non-empty open set (which all contain 0) intersects $[a, b]$, so every point of \mathbb{R} is adherent to $[a, b]$. Thus, $\overline{[a, b]} = \mathbb{R}$.
- *Boundary:* $\text{Fr}([a, b]) = \mathbb{R} \setminus] - \min(|a|, b), \min(|a|, b)[$.

Case 2: $0 \notin [a, b]$ (so either $a > 0$ or $b < 0$). By symmetry, assume $a > 0$.

- *Interior:* No non-empty symmetric open set $] - \alpha, \alpha[$ can be contained in $[a, b] \subset]0, \infty[$ because $] - \alpha, \alpha[$ always contains negative numbers. Thus, $[a, b]^\circ = \emptyset$.
- *Closure:* The closure is the smallest closed set containing $[a, b]$. The closed sets are $] - \infty, -\alpha] \cup [\alpha, \infty[$. To contain $[a, b] \subset]0, \infty[$, we need $\alpha \leq a$. The smallest (for inclusion) such closed set is obtained for the largest possible α , which is $\alpha = a$. Therefore,

$$\overline{[a, b]} =] - \infty, -a] \cup [a, \infty[.$$

(If $b < 0$, the result is $\overline{[a, b]} =] - \infty, b] \cup [-b, \infty[$.)

- *Boundary:* Since the interior is empty, the boundary is the closure itself.

Exercise 14. Let (E, \mathcal{T}) be a topological space and let $A, B \subseteq E$.

1. Show that $\text{Fr}(A \cup B) \subseteq \text{Fr}(A) \cup \text{Fr}(B)$, and that equality holds if A and B are disjoint. Give an example in \mathbb{R} (with its standard topology) where the inclusion is strict.
2. Show that $\text{Fr}(\overline{A}) \subseteq \text{Fr}(A)$ and $\text{Fr}(\overset{\circ}{A}) \subseteq \text{Fr}(A)$.

Solution 14. We recall that the boundary of a set S is $\text{Fr}(S) = \overline{S} \setminus \overset{\circ}{S}$.

1. **Proof of inclusion.** Let $x \in \text{Fr}(A \cup B)$. Then $x \in \overline{A \cup B} = \overline{A} \cup \overline{B}$, so $x \in \overline{A}$ or $x \in \overline{B}$.

Suppose $x \notin \text{Fr}(A) \cup \text{Fr}(B)$. Then x must be an interior point of both A and B (since it is in their closures but not on their boundaries). This implies $x \in \overset{\circ}{A} \cap \overset{\circ}{B} \subseteq A \overset{\circ}{\cup} B$. But then $x \notin \text{Fr}(A \cup B)$, a contradiction. Therefore, $x \in \text{Fr}(A) \cup \text{Fr}(B)$.

Equality for disjoint sets. Now assume $A \cap B = \emptyset$. We already have $\text{Fr}(A \cup B) \subseteq \text{Fr}(A) \cup \text{Fr}(B)$. For the reverse inclusion, let $x \in \text{Fr}(A)$ (the argument for $\text{Fr}(B)$ is identical).

Since $x \in \overline{A} \subseteq \overline{A \cup B}$, we need to show $x \notin A \overset{\circ}{\cup} B$. Suppose for contradiction that $x \in A \overset{\circ}{\cup} B$. Then there is an open neighborhood U of x such that $U \subseteq A \cup B$. Since $x \in \text{Fr}(A)$, every neighborhood of x intersects A^c , so $U \cap A^c \neq \emptyset$, which implies $U \cap B \neq \emptyset$ (because $U \subseteq A \cup B$). Thus, $x \in \overline{B}$. But since A and B are disjoint and $x \in \overline{A}$, if x were also in B , it would be a limit point of A inside B , which is allowed. However, because $x \in \text{Fr}(A)$, it is not in $\overset{\circ}{A}$. The key is that U cannot be contained in A , but our assumption that $U \subseteq A \cup B$ with $U \cap B \neq \emptyset$ doesn't immediately give a contradiction.

A cleaner argument: For disjoint A, B , we have $A \overset{\circ}{\cup} B = \overset{\circ}{A} \cup \overset{\circ}{B}$ (this is a standard result). Then

$$\text{Fr}(A \cup B) = \overline{A \cup B} \setminus A \overset{\circ}{\cup} B = (\overline{A} \cup \overline{B}) \setminus (\overset{\circ}{A} \cup \overset{\circ}{B}) = (\overline{A} \setminus \overset{\circ}{A}) \cup (\overline{B} \setminus \overset{\circ}{B}) = \text{Fr}(A) \cup \text{Fr}(B).$$

Counterexample for strict inclusion. Let $A = [0, 1]$ and $B = [1, 2]$ in \mathbb{R} . Then $A \cup B = [0, 2]$.

$$\text{Fr}(A \cup B) = \{0, 2\}, \quad \text{Fr}(A) = \{0, 1\}, \quad \text{Fr}(B) = \{1, 2\}.$$

Thus, $\text{Fr}(A) \cup \text{Fr}(B) = \{0, 1, 2\} \supsetneq \{0, 2\} = \text{Fr}(A \cup B)$.

2. The original statement " $\text{Fr}(A) \subseteq \text{Fr}(A)$ " is a tautology and likely contains a typo. The intended and correct statements are the ones given in the exercise above.

Proof that $\text{Fr}(\bar{A}) \subseteq \text{Fr}(A)$. We have $\bar{\bar{A}} = \bar{A}$. Also, $\overset{\circ}{\bar{A}} \supseteq \overset{\circ}{A}$. Therefore,

$$\text{Fr}(\bar{A}) = \bar{A} \setminus \overset{\circ}{\bar{A}} \subseteq \bar{A} \setminus \overset{\circ}{A} = \text{Fr}(A).$$

Proof that $\text{Fr}(\overset{\circ}{A}) \subseteq \text{Fr}(A)$. We have $\bar{\overset{\circ}{A}} \subseteq \bar{A}$. Also, $\overset{\circ}{\overset{\circ}{A}} = \overset{\circ}{A}$. Therefore,

$$\text{Fr}(\overset{\circ}{A}) = \bar{\overset{\circ}{A}} \setminus \overset{\circ}{\overset{\circ}{A}} \subseteq \bar{A} \setminus \overset{\circ}{A} = \text{Fr}(A).$$

Exercise 15. Let (E, \mathcal{T}) be a topological space.

1. Show that $A \subseteq B \implies \bar{A} \subseteq \bar{B}$ for any subsets $A, B \subseteq E$.
2. Deduce that $\bigcup_{i \in I} \bar{A}_i \subseteq \overline{\bigcup_{i \in I} A_i}$, and show that equality holds when the index set I is finite.
3. Give an example in \mathbb{R} (with its standard topology) where this inclusion is strict.
4. Show that $\overline{\bigcap_{i \in I} A_i} \subseteq \bigcap_{i \in I} \bar{A}_i$ and give an example in \mathbb{R} where the inclusion $\overline{A \cap B} \subseteq \bar{A} \cap \bar{B}$ is strict.
5. State the corresponding results for interiors.

Solution 15. 1. Suppose $A \subseteq B$. The closure \bar{B} is a closed set containing B , and hence contains A . Since \bar{A} is the smallest closed set containing A , it must be that $\bar{A} \subseteq \bar{B}$.

2. For each $j \in I$, $A_j \subseteq \bigcup_{i \in I} A_i$, so by part 1, $\bar{A}_j \subseteq \overline{\bigcup_{i \in I} A_i}$. Taking the union over all $j \in I$ gives the inclusion $\bigcup_{i \in I} \bar{A}_i \subseteq \overline{\bigcup_{i \in I} A_i}$.

Now assume $I = \{1, \dots, n\}$ is finite. We will show the reverse inclusion $\overline{\bigcup_{k=1}^n A_k} \subseteq \bigcup_{k=1}^n \bar{A}_k$. The right-hand side is a finite union of closed sets, hence closed. It also contains $\bigcup_{k=1}^n A_k$. Since the closure $\overline{\bigcup_{k=1}^n A_k}$ is the smallest closed set with this property, the inclusion follows.

3. Let $A_n = \left\{ \frac{1}{n} \right\}$ for $n \in \mathbb{N}^*$. Then $\bar{A}_n = A_n$, so

$$\bigcup_{n=1}^{\infty} \bar{A}_n = \left\{ 1, \frac{1}{2}, \frac{1}{3}, \dots \right\}.$$

However, $\bigcup_{n=1}^{\infty} A_n = \left\{ 1, \frac{1}{2}, \frac{1}{3}, \dots \right\}$, and its closure is

$$\overline{\bigcup_{n=1}^{\infty} A_n} = \left\{ 1, \frac{1}{2}, \frac{1}{3}, \dots \right\} \cup \{0\}.$$

Thus, the inclusion is strict.

4. Let $x \in \overline{\bigcap_{i \in I} A_i}$. Then every neighborhood of x intersects $\bigcap_{i \in I} A_i$, and therefore intersects every A_i . This means $x \in \bar{A}_i$ for all $i \in I$, so $x \in \bigcap_{i \in I} \bar{A}_i$.

For a strict example, let $A =]0, 1[$ and $B =]1, 2[$ in \mathbb{R} . Then $A \cap B = \emptyset$, so $\overline{A \cap B} = \emptyset$. However, $\bar{A} = [0, 1]$ and $\bar{B} = [1, 2]$, so $\bar{A} \cap \bar{B} = \{1\} \neq \emptyset$.

5. The corresponding results for interiors (denoted by $\overset{\circ}{\cdot}$) are:

- $A \subseteq B \implies \overset{\circ}{A} \subseteq \overset{\circ}{B}$.
- $\bigcap_{i \in I} \overset{\circ}{A}_i = \overset{\circ}{\bigcap_{i \in I} A_i}$.
- $\bigcup_{i \in I} \overset{\circ}{A}_i \supseteq \overset{\circ}{\bigcup_{i \in I} A_i}$, with equality if I is finite.

These follow from the duality between interior and closure (via complements) or can be proven directly using the definition of interior as the largest open set contained in a given set.

1.6 Dense Subsets

Definition 1.29. Let (E, \mathcal{T}) be a topological space and let $D \subseteq E$. The set D is said to be *dense* in E if its closure is the entire space, i.e., $\overline{D} = E$.

The following proposition provides a very useful characterization of dense sets.

Proposition 1.30. A subset $D \subseteq E$ is dense in E if and only if every nonempty open set in E intersects D .

Proof. (\implies) Suppose D is dense, so $\overline{D} = E$. Let O be a nonempty open set. If $O \cap D = \emptyset$, then $O \subseteq E \setminus D$. Since O is open, its complement $E \setminus O$ is closed and contains D . Therefore, $\overline{D} \subseteq E \setminus O$, which implies $E \subseteq E \setminus O$, a contradiction since O is nonempty.

(\impliedby) Conversely, assume every nonempty open set meets D . Suppose $\overline{D} \neq E$. Then the set $O = E \setminus \overline{D}$ is a nonempty open set (as the complement of a proper closed set) that is disjoint from D , contradicting our hypothesis. Hence, $\overline{D} = E$. ■

Theorem 1.31. 1. The set of rational numbers \mathbb{Q} is dense in \mathbb{R} (i.e., $\overline{\mathbb{Q}} = \mathbb{R}$). Every nonempty open interval in \mathbb{R} contains infinitely many rational numbers.

2. The set of irrational numbers $\mathbb{R} \setminus \mathbb{Q}$ is dense in \mathbb{R} (i.e., $\overline{\mathbb{R} \setminus \mathbb{Q}} = \mathbb{R}$). Every nonempty open interval in \mathbb{R} contains infinitely many irrational numbers.

Proof. Let $I \subseteq \mathbb{R}$ be a nonempty open interval. Then I contains an open interval of the form $]x, y[$ with $x, y \in \mathbb{R}$. We may therefore assume $I =]x, y[$.

1. I contains a rational number. We prove that for all $x, y \in \mathbb{R}$ with $x < y$, there exists a rational number $r \in \mathbb{Q}$ such that $x < r < y$.

By the Archimedean property, there exists $q \in \mathbb{N}^*$ such that $q > 1/(y-x)$, so that $q(y-x) > 1$. The interval $]qx, qy[$ has length greater than 1, so it must contain an integer p . Therefore, $qx < p < qy$, which implies $x < p/q < y$. The number $r = p/q$ is the desired rational.

2. I contains an irrational number. Consider the real numbers $x - \sqrt{2}$ and $y - \sqrt{2}$. Since $x < y$, we have $x - \sqrt{2} < y - \sqrt{2}$. By part 1, there exists a rational number $r \in \mathbb{Q}$ such that

$$x - \sqrt{2} < r < y - \sqrt{2}.$$

Adding $\sqrt{2}$ to all parts of the inequality yields $x < r + \sqrt{2} < y$. The number $\alpha = r + \sqrt{2}$ is irrational, for if it were rational, then $\sqrt{2} = \alpha - r$ would be rational as the difference of two rationals, which is a contradiction. Thus, I contains the irrational number α . ■

Exercise 16. Consider \mathbb{R} equipped with its standard topology. Determine the interior and the closure of \mathbb{Q} and $\mathbb{R} \setminus \mathbb{Q}$.

Solution 16. We will use the fundamental fact that both \mathbb{Q} and $\mathbb{R} \setminus \mathbb{Q}$ are dense in \mathbb{R} .

For \mathbb{Q} :

- *Closure:* Since \mathbb{Q} is dense in \mathbb{R} , its closure is the entire space:

$$\overline{\mathbb{Q}} = \mathbb{R}.$$

- *Interior:* We claim $\overset{\circ}{\mathbb{Q}} = \emptyset$. To see this, suppose for contradiction that there exists a nonempty open set $U \subseteq \mathbb{Q}$. In the standard topology on \mathbb{R} , every nonempty open set contains an open interval $]a, b[$. However, any such interval is uncountable, while \mathbb{Q} is countable, so $]a, b[\not\subseteq \mathbb{Q}$. Therefore, no nonempty open set is contained in \mathbb{Q} , and its interior is empty.

For $\mathbb{R} \setminus \mathbb{Q}$:

- *Closure:* The set of irrationals is also dense in \mathbb{R} . Hence,

$$\overline{\mathbb{R} \setminus \mathbb{Q}} = \mathbb{R}.$$

- *Interior:* The argument is analogous to that for \mathbb{Q} . Any nonempty open interval $]a, b[$ in \mathbb{R} contains rational numbers (since \mathbb{Q} is dense). Therefore, no nonempty open interval can be contained in $\mathbb{R} \setminus \mathbb{Q}$, which implies that its interior is also empty:

$$\mathbb{R} \setminus \overset{\circ}{\mathbb{Q}} = \emptyset.$$

In summary:

$$\overline{\mathbb{Q}} = \mathbb{R}, \quad \overset{\circ}{\mathbb{Q}} = \emptyset, \quad \overline{\mathbb{R} \setminus \mathbb{Q}} = \mathbb{R}, \quad \mathbb{R} \setminus \overset{\circ}{\mathbb{Q}} = \emptyset.$$

Exercise 17. Show that a subset A of a topological space (E, \mathcal{T}) has nonempty interior if and only if it intersects every dense subset of E .

Solution 17. We will prove both implications.

(\Rightarrow) Assume that $\overset{\circ}{A} \neq \emptyset$. Let $D \subseteq E$ be an arbitrary dense subset, so that $\overline{D} = E$.

Since $\overset{\circ}{A}$ is a nonempty open set and D is dense, Proposition 1.30 tells us that D must intersect every nonempty open set. In particular, $D \cap \overset{\circ}{A} \neq \emptyset$. Since $\overset{\circ}{A} \subseteq A$, it follows that $D \cap A \neq \emptyset$.

(\Leftarrow) We prove the contrapositive. Suppose that $\overset{\circ}{A} = \emptyset$. We will construct a dense subset D of E that is disjoint from A .

Consider the complement of A , $D := E \setminus A$. We claim that D is dense in E . To see this, note that the closure of D is

$$\overline{D} = \overline{E \setminus A} = E \setminus \overset{\circ}{A}.$$

Since $\overset{\circ}{A} = \emptyset$, we have $\overline{D} = E \setminus \emptyset = E$. Therefore, D is dense. By construction, $D \cap A = (E \setminus A) \cap A = \emptyset$.

Thus, if A has empty interior, there exists a dense set (namely, its complement) that it does not meet. This completes the proof of the contrapositive and hence the entire statement.

Exercise 18. Let (E, \mathcal{T}) be a topological space, and let A and B be two dense subsets of E . Suppose further that A is open in E . Show that $A \cap B$ is dense in E . (You may use Proposition 1.30: a set is dense if and only if it intersects every nonempty open set).

Solution 18. We will use the characterization of dense sets from Proposition 1.30: a subset $S \subseteq E$ is dense if and only if $S \cap U \neq \emptyset$ for every nonempty open set $U \subseteq E$.

Let U be an arbitrary nonempty open subset of E . We must show that $(A \cap B) \cap U \neq \emptyset$.

Since A is open and U is open, their intersection $A \cap U$ is also open. Moreover, because A is dense in E , it intersects every nonempty open set. In particular, $A \cap U \neq \emptyset$.

Now, $A \cap U$ is a nonempty open set, and B is a dense subset of E . By the same characterization, B must intersect every nonempty open set, so

$$B \cap (A \cap U) \neq \emptyset.$$

This is equivalent to $(A \cap B) \cap U \neq \emptyset$.

Since U was an arbitrary nonempty open set, we conclude that $A \cap B$ intersects every nonempty open set in E . Therefore, by Proposition 1.30, $A \cap B$ is dense in E .

1.7 Hausdorff Spaces

Definition 1.32. A topological space (E, \mathcal{T}) is said to be *Hausdorff* (or *separated*) if for every pair of distinct points $x, y \in E$, there exist neighborhoods $V \in \mathcal{V}(x)$ and $W \in \mathcal{V}(y)$ such that $V \cap W = \emptyset$.

Example 1.33. 1. If E is equipped with the discrete topology, then for any distinct $x, y \in E$, the singletons $\{x\}$ and $\{y\}$ are disjoint open sets. Hence, E is Hausdorff.

2. If E is equipped with the trivial topology and contains at least two distinct points, then E is not Hausdorff. Indeed, for any $x \in E$, the only neighborhood of x is E itself, so any two neighborhoods of distinct points must intersect.

3. The set of real numbers \mathbb{R} with its standard topology is a Hausdorff space. Indeed, for any distinct $x, y \in \mathbb{R}$, let $r = |x - y|/3 > 0$. The open intervals $]x - r, x + r[$ and $]y - r, y + r[$ are disjoint neighborhoods of x and y , respectively.

Proposition 1.34. If (E, \mathcal{T}) is a Hausdorff space, then for every point $\ell \in E$, we have

$$\bigcap_{V \in \mathcal{V}(\ell)} V = \{\ell\}.$$

Proof. It is clear that $\ell \in V$ for every $V \in \mathcal{V}(\ell)$, so $\ell \in \bigcap_{V \in \mathcal{V}(\ell)} V$.

Conversely, let $y \in E \setminus \{\ell\}$. Since E is Hausdorff, there exists a neighborhood $V \in \mathcal{V}(\ell)$ such that $y \notin V$. Therefore, $y \notin \bigcap_{V \in \mathcal{V}(\ell)} V$.

This shows that $\bigcap_{V \in \mathcal{V}(\ell)} V \subseteq \{\ell\}$, completing the proof. ■

Exercise 19. Let $E :=]0, +\infty[$. For every $\alpha \in \mathbb{R}$, define $O_\alpha :=]\alpha, +\infty[$. Consider the family \mathcal{T} of subsets of E given by

$$\mathcal{T} = \{\emptyset, E\} \cup \{O_\alpha \cap E \mid \alpha \in \mathbb{R}\}.$$

1. Show that \mathcal{T} constitutes a topology on E .
2. Determine the closed sets of the topological space (E, \mathcal{T}) .
3. Give (without proof) the interior $\overset{\circ}{A}$ and the closure \bar{A} for the following sets:

$$A = [4, 9[, \quad A = [-2, +\infty[, \quad A = \{2, 3, 4\}, \quad A = \mathbb{N}.$$

(Note: All sets are considered as subsets of $E =]0, +\infty[$).

4. Show that (E, \mathcal{T}) is not a Hausdorff space.

Solution 19. First, we note that the open sets in \mathcal{T} are precisely \emptyset , E , and all intervals of the form $] \alpha, +\infty[$ where $\alpha \geq 0$. (If $\alpha < 0$, then $O_\alpha \cap E =]0, +\infty[= E$).

1. We verify the axioms of a topology.

- $\emptyset, E \in \mathcal{T}$ by definition.
- Let $\{U_i\}_{i \in I} \subseteq \mathcal{T}$ be an arbitrary collection of open sets. Each U_i is either \emptyset , E , or $] \alpha_i, +\infty[$ for some $\alpha_i \geq 0$. Their union is $] \inf_{i \in I} \alpha_i, +\infty[$ (with the convention that $\inf\{\text{set containing } E\} = 0$), which is in \mathcal{T} .
- Let $U_1 =] \alpha, +\infty[$ and $U_2 =] \beta, +\infty[$ be two non-trivial open sets. Their intersection is $] \max(\alpha, \beta), +\infty[\in \mathcal{T}$. The intersection with \emptyset or E is trivial.

Thus, \mathcal{T} is a topology on E , known as the *right-ray topology*.

2. The closed sets are the complements of the open sets. They are:

$$E, \quad \emptyset, \quad \text{and all sets of the form }]0, \alpha] \text{ for } \alpha \in [0, +\infty[.$$

3. We analyze each set as a subset of $E =]0, +\infty[$.

- $A = [4, 9[$: The largest open set contained in A is empty, so $\overset{\circ}{A} = \emptyset$. The smallest closed set containing A is $]0, 9]$, so $\overline{A} =]0, 9]$.
- $A = [-2, +\infty[$: Since we are in E , this is just E itself. So $\overset{\circ}{A} = E$ and $\overline{A} = E$.
- $A = \{2, 3, 4\}$: No non-empty open set is contained in a finite set, so $\overset{\circ}{A} = \emptyset$. The smallest closed set containing A is $]0, 4]$, so $\overline{A} =]0, 4]$.
- $A = \mathbb{N}$: Again, $\overset{\circ}{A} = \emptyset$. Since \mathbb{N} is unbounded above in E , the only closed set containing it is E itself. Thus, $\overline{A} = E$.

4. To show (E, \mathcal{T}) is not Hausdorff, take any two distinct points $x, y \in E$ with $x < y$. Every non-empty open neighborhood of x is of the form $]a, +\infty[$ with $a < x$, and every non-empty open neighborhood of y is of the form $]b, +\infty[$ with $b < y$. Since $x < y$, we can choose $a = b = x/2$, but more importantly, *any* neighborhood of x is $]a, +\infty[$ with $a < x < y$, which necessarily contains y . Similarly, any neighborhood of y is $]b, +\infty[$ with $b < y$, which contains x if $b < x$.

In fact, for any two points $x < y$, every neighborhood of x contains y . Therefore, there do not exist disjoint neighborhoods of x and y , so the space is not Hausdorff.

1.8 Induced and Product Topologies

1.8.1 Induced Topology

Definition 1.35. Let (E, \mathcal{T}) be a topological space and let $A \subseteq E$. The family \mathcal{T}_A of subsets of A defined by

$$\mathcal{T}_A = \{U \subseteq A \mid \exists O \in \mathcal{T} \text{ such that } U = O \cap A\}$$

is a topology on A , called the *subspace topology* (or *induced topology*) on A . The topological space (A, \mathcal{T}_A) is called a *subspace* of E .

Proof. We verify the axioms of a topology for (A, \mathcal{T}_A) .

1. Since $\emptyset, E \in \mathcal{T}$, we have $\emptyset = \emptyset \cap A \in \mathcal{T}_A$ and $A = E \cap A \in \mathcal{T}_A$.
2. Let $U_1, U_2 \in \mathcal{T}_A$. Then there exist $O_1, O_2 \in \mathcal{T}$ such that $U_i = O_i \cap A$ for $i = 1, 2$. Their intersection is

$$U_1 \cap U_2 = (O_1 \cap O_2) \cap A.$$

Since \mathcal{T} is a topology, $O_1 \cap O_2 \in \mathcal{T}$, so $U_1 \cap U_2 \in \mathcal{T}_A$.

3. Let $\{U_i\}_{i \in I} \subseteq \mathcal{T}_A$. For each i , there is an $O_i \in \mathcal{T}$ with $U_i = O_i \cap A$. Their union is

$$\bigcup_{i \in I} U_i = \left(\bigcup_{i \in I} O_i \right) \cap A.$$

Since $\bigcup_{i \in I} O_i \in \mathcal{T}$, the union is in \mathcal{T}_A . ■

Proposition 1.36. Let (E, \mathcal{T}) be a topological space and $A \subseteq E$.

1. The closed sets of the subspace (A, \mathcal{T}_A) are precisely the sets of the form $F \cap A$, where F is a closed set in E .
2. The neighborhoods of a point $a \in A$ in the subspace topology are precisely the sets of the form $V \cap A$, where V is a neighborhood of a in E .

Example 1.37. Consider the interval $[0, 2]$ as a subspace of \mathbb{R} with its standard topology. The set $[0, 1[$ is open in $[0, 2]$ because $[0, 1[=] - 1, 1[\cap [0, 2]$ and $] - 1, 1[$ is open in \mathbb{R} . Similarly, $[0, 1[$ is closed in $[-1, 1]$ because $[0, 1[= [0, 3] \cap [-1, 1]$ and $[0, 3]$ is closed in \mathbb{R} . However, $[0, 1[$ is neither open nor closed in \mathbb{R} itself.

Proposition 1.38 (Transitivity of the Induced Topology). Let (E, \mathcal{T}) be a topological space, and let $B \subseteq A \subseteq E$. The subspace topology induced on B directly from E is the same as the topology obtained by first inducing the topology on A from E and then inducing the topology on B from A .

1.8.2 Product Topology

Let (E_1, \mathcal{T}_1) and (E_2, \mathcal{T}_2) be two topological spaces.

Definition 1.39. An *elementary open set* in the Cartesian product $E_1 \times E_2$ is a set of the form $U_1 \times U_2$, where $U_1 \in \mathcal{T}_1$ and $U_2 \in \mathcal{T}_2$. The *product topology* on $E_1 \times E_2$ is the topology whose open sets are arbitrary unions of elementary open sets.

Remark 1.40. This defines a topology because:

- $E_1 \times E_2$ and $\emptyset = \emptyset \times \emptyset$ are elementary open sets.
- The intersection of two elementary open sets is elementary: $(U_1 \times U_2) \cap (V_1 \times V_2) = (U_1 \cap V_1) \times (U_2 \cap V_2)$.
- Arbitrary unions of unions of elementary sets are again unions of elementary sets.

Definition 1.41. The *standard topology* on \mathbb{R}^n is the product topology obtained by taking the product of n copies of \mathbb{R} with its standard topology.

Example 1.42. The standard topology on \mathbb{R}^2 has a basis consisting of open rectangles, i.e., sets of the form $]a, b[\times]c, d[$.

Proposition 1.43 (Transitivity of the Induced Topology). Let (E, \mathcal{T}) be a topological space, and let $B \subseteq A \subseteq E$ be subsets of E . The topology induced on B directly by \mathcal{T} is identical to the topology obtained by first inducing the topology on A from \mathcal{T} and then inducing the topology on B from A .

Proof. Let \mathcal{T}_B^E denote the topology on B induced directly from E , and let \mathcal{T}_B^A denote the topology on B induced from the subspace A (which itself has the topology \mathcal{T}_A induced from E).

By definition,

$$\mathcal{T}_B^E = \{U \subseteq B \mid \exists O \in \mathcal{T} \text{ such that } U = O \cap B\}.$$

For \mathcal{T}_B^A , a set $U \subseteq B$ is open if and only if there exists an open set V in the subspace A such that $U = V \cap B$. But V is open in A if and only if there exists $O \in \mathcal{T}$ such that $V = O \cap A$. Therefore,

$$U = V \cap B = (O \cap A) \cap B = O \cap (A \cap B) = O \cap B,$$

since $B \subseteq A$. This shows that $\mathcal{T}_B^A = \mathcal{T}_B^E$. ■

1.8.3 Product Topology

Let (E_1, \mathcal{T}_1) and (E_2, \mathcal{T}_2) be two topological spaces.

Definition 1.44. An *elementary open set* in the Cartesian product $E_1 \times E_2$ is a subset $W \subseteq E_1 \times E_2$ of the form $W = O_1 \times O_2$, where $O_1 \in \mathcal{T}_1$ and $O_2 \in \mathcal{T}_2$. The *product topology* on $E_1 \times E_2$ is the topology whose open sets are arbitrary unions of elementary open sets.

Remark 1.45. This indeed defines a topology because:

- $E_1 \times E_2$ and $\emptyset = \emptyset \times \emptyset$ are elementary open sets.
- The intersection of two elementary open sets is elementary:

$$(O_1 \times O_2) \cap (O'_1 \times O'_2) = (O_1 \cap O'_1) \times (O_2 \cap O'_2).$$

- Arbitrary unions of unions of elementary sets are unions of elementary sets.

Definition 1.46. The *standard topology* on \mathbb{R}^n is the topology obtained by taking the n -fold product of \mathbb{R} with its standard topology.

Example 1.47. The standard topology on \mathbb{R}^2 has a basis consisting of open rectangles, i.e., sets of the form $]a, b[\times]c, d[$.

Proposition 1.48. Let $x = (x_1, x_2) \in E_1 \times E_2$.

1. The collection $\{V_1 \times V_2 \mid V_1 \in \mathcal{V}(x_1), V_2 \in \mathcal{V}(x_2)\}$ is a neighborhood basis for x in $E_1 \times E_2$.
2. More generally, if $\mathcal{B}(x_1)$ (resp. $\mathcal{B}(x_2)$) is a neighborhood basis for x_1 in E_1 (resp. for x_2 in E_2), then the collection $\{B_1 \times B_2 \mid B_1 \in \mathcal{B}(x_1), B_2 \in \mathcal{B}(x_2)\}$ is a neighborhood basis for x in $E_1 \times E_2$.

Proof. We prove the second, more general statement; the first follows as a special case.

Let W be an arbitrary neighborhood of $x = (x_1, x_2)$ in the product space $E_1 \times E_2$. By definition of the product topology, there exists an elementary open set $O_1 \times O_2$ such that

$$x \in O_1 \times O_2 \subseteq W,$$

where O_1 is open in E_1 and O_2 is open in E_2 . Since O_1 is an open neighborhood of x_1 , there exists $B_1 \in \mathcal{B}(x_1)$ such that $B_1 \subseteq O_1$. Similarly, there exists $B_2 \in \mathcal{B}(x_2)$ such that $B_2 \subseteq O_2$.

It follows that $B_1 \times B_2 \subseteq O_1 \times O_2 \subseteq W$, and $B_1 \times B_2$ is an element of the proposed neighborhood basis. This shows that every neighborhood of x contains a set from the collection $\{B_1 \times B_2\}$, which is precisely the definition of a neighborhood basis. ■

Example 1.49. Let \mathbb{R}^2 be equipped with its standard topology, and let $x = (x_1, x_2) \in \mathbb{R}^2$. The family

$$\left\{]x_1 - \varepsilon, x_1 + \varepsilon[\times]x_2 - \varepsilon, x_2 + \varepsilon[\mid \varepsilon \in \mathbb{R}_+^* \right\}$$

is a fundamental system of neighborhoods (neighborhood basis) for x .

Proposition 1.50. If E_1 and E_2 are Hausdorff spaces, then their product $E_1 \times E_2$ (with the product topology) is also Hausdorff.

Proof. Let (x_1, x_2) and (y_1, y_2) be two distinct points in $E_1 \times E_2$. Then either $x_1 \neq y_1$ or $x_2 \neq y_2$ (or both).

Suppose $x_1 \neq y_1$. Since E_1 is Hausdorff, there exist disjoint open sets $U_1, V_1 \subseteq E_1$ such that $x_1 \in U_1$ and $y_1 \in V_1$. The sets $U_1 \times E_2$ and $V_1 \times E_2$ are open in the product topology, contain (x_1, x_2) and (y_1, y_2) respectively, and are disjoint because $U_1 \cap V_1 = \emptyset$.

The case $x_2 \neq y_2$ is analogous. Therefore, $E_1 \times E_2$ is Hausdorff. ■

Example 1.51. By induction and Proposition 1.50, the standard topology on \mathbb{R}^n for any $n \in \mathbb{N}^*$ is Hausdorff.

Exercise 20. Let (E_1, \mathcal{T}_1) and (E_2, \mathcal{T}_2) be topological spaces, and let $E = E_1 \times E_2$ be their product space. Show that E is Hausdorff if and only if both E_1 and E_2 are Hausdorff.

Solution 20. (\Leftarrow) This is precisely the statement of Proposition 1.50.

(\Rightarrow) Suppose $E_1 \times E_2$ is Hausdorff. We will show E_1 is Hausdorff; the proof for E_2 is identical.

Let $x_1, y_1 \in E_1$ with $x_1 \neq y_1$. Fix an arbitrary point $z_2 \in E_2$. Then (x_1, z_2) and (y_1, z_2) are distinct points in $E_1 \times E_2$. Since the product is Hausdorff, there exist disjoint open sets $W, W' \subseteq E_1 \times E_2$ such that $(x_1, z_2) \in W$ and $(y_1, z_2) \in W'$.

By the definition of the product topology, there exist elementary open sets $U_1 \times U_2 \subseteq W$ and $V_1 \times V_2 \subseteq W'$ containing (x_1, z_2) and (y_1, z_2) respectively. Therefore, $x_1 \in U_1$, $y_1 \in V_1$, and $(U_1 \times U_2) \cap (V_1 \times V_2) = \emptyset$, which implies $U_1 \cap V_1 = \emptyset$. Thus, U_1 and V_1 are disjoint open neighborhoods of x_1 and y_1 in E_1 , so E_1 is Hausdorff.

Exercise 21. Closure, Interior, and Boundary of a Product Let E_1 and E_2 be topological spaces, let $A \subseteq E_1$, $B \subseteq E_2$, and let $C = A \times B$ be a subset of the product space $E = E_1 \times E_2$. Show that:

1. $\overline{C} = \overline{A} \times \overline{B}$,
2. $\overset{\circ}{C} = \overset{\circ}{A} \times \overset{\circ}{B}$,
3. $\text{Fr}(C) = (\text{Fr}(A) \times \overline{B}) \cup (\overline{A} \times \text{Fr}(B))$.

Solution 21. 1. **Closure.** (\supseteq) Let $(x, y) \in \overline{A} \times \overline{B}$. Let W be an arbitrary open neighborhood of (x, y) in $E_1 \times E_2$. Then W contains an elementary open set $U \times V$ with $x \in U$ and $y \in V$. Since $x \in \overline{A}$, $U \cap A \neq \emptyset$; since $y \in \overline{B}$, $V \cap B \neq \emptyset$. Therefore, $(U \times V) \cap (A \times B) \neq \emptyset$, so $W \cap C \neq \emptyset$. Hence, $(x, y) \in \overline{C}$.

(\subseteq) Let $(x, y) \in \overline{C}$. Suppose $x \notin \overline{A}$. Then there exists an open set $U \subseteq E_1$ with $x \in U$ and $U \cap A = \emptyset$. Then $U \times E_2$ is an open neighborhood of (x, y) disjoint from C , contradicting $(x, y) \in \overline{C}$. Thus, $x \in \overline{A}$. Similarly, $y \in \overline{B}$. Therefore, $\overline{C} \subseteq \overline{A} \times \overline{B}$.

2. **Interior.** (\subseteq) $\overset{\circ}{A} \times \overset{\circ}{B}$ is an open set (as a product of open sets) contained in $A \times B = C$. Therefore, $\overset{\circ}{A} \times \overset{\circ}{B} \subseteq \overset{\circ}{C}$.

(\supseteq) Let $(x, y) \in \overset{\circ}{C}$. Then there exists an elementary open set $U \times V \subseteq C = A \times B$ with $x \in U$ and $y \in V$. This implies $U \subseteq A$ and $V \subseteq B$, so $x \in \overset{\circ}{A}$ and $y \in \overset{\circ}{B}$. Hence, $(x, y) \in \overset{\circ}{A} \times \overset{\circ}{B}$.

3. **Boundary.** By definition, $\text{Fr}(C) = \overline{C} \setminus \overset{\circ}{C}$. Using parts 1 and 2, this is

$$(\overline{A} \times \overline{B}) \setminus (\overset{\circ}{A} \times \overset{\circ}{B}).$$

A point (x, y) is in this set if and only if $(x \in \overline{A}$ and $y \in \overline{B})$ and *not* $(x \in \overset{\circ}{A}$ and $y \in \overset{\circ}{B})$. By De Morgan's law, this is equivalent to:

$$(x \in \overline{A} \text{ and } y \in \overline{B} \text{ and } x \notin \overset{\circ}{A}) \quad \text{or} \quad (x \in \overline{A} \text{ and } y \in \overline{B} \text{ and } y \notin \overset{\circ}{B}).$$

This is precisely

$$(\text{Fr}(A) \times \overline{B}) \cup (\overline{A} \times \text{Fr}(B)),$$

since $\text{Fr}(A) = \overline{A} \setminus \overset{\circ}{A}$ and $\text{Fr}(B) = \overline{B} \setminus \overset{\circ}{B}$.

1.9 Convergent Sequences

Throughout this section, (E, \mathcal{T}) denotes a topological space.

Definition 1.52. Let $(x_n)_{n \in \mathbb{N}}$ be a sequence of elements of E and let $\ell \in E$. We say that the sequence (x_n) *converges* to ℓ (or *tends* to ℓ) as $n \rightarrow +\infty$ if

$$\forall V \in \mathcal{V}(\ell), \exists n_0 = n_0(V) \in \mathbb{N} \text{ such that } \forall n \in \mathbb{N}, n \geq n_0 \implies x_n \in V.$$

If (x_n) converges to ℓ , we say that ℓ is the *limit* of the sequence (x_n) .

Remark 1.53. 1. A sequence in a general topological space may have multiple limits. For example, if \mathcal{T} is the trivial topology on E , then every point of E is a limit of every sequence in E .

2. The convergent sequences in a discrete space are precisely the eventually constant (or *stationary*) sequences.

Theorem 1.54. In a Hausdorff space, the limit of any convergent sequence is unique.

Proof. Suppose, for contradiction, that a sequence $(x_n)_{n \in \mathbb{N}}$ in a Hausdorff space (E, \mathcal{T}) has two distinct limits ℓ_1 and ℓ_2 . Since E is Hausdorff, there exist disjoint neighborhoods $V \in \mathcal{V}(\ell_1)$ and $W \in \mathcal{V}(\ell_2)$ such that $V \cap W = \emptyset$.

Because ℓ_1 is a limit, there exists $n_1 \in \mathbb{N}$ such that $x_n \in V$ for all $n \geq n_1$. Similarly, because ℓ_2 is a limit, there exists $n_2 \in \mathbb{N}$ such that $x_n \in W$ for all $n \geq n_2$. Let $n = \max(n_1, n_2)$. Then $x_n \in V \cap W$, which contradicts $V \cap W = \emptyset$. Therefore, the limit must be unique. ■

Definition 1.55. A sequence $(x_n)_{n \in \mathbb{N}}$ in a topological space E is said to be *convergent* (in E) if there exists $\ell \in E$ such that $x_n \rightarrow \ell$ as $n \rightarrow +\infty$. Such a point ℓ is called a *limit point* of the sequence (x_n) in E .

Remark 1.56. If $(y_n)_{n \in \mathbb{N}}$ is a sequence in a subset $A \subseteq E$ and (y_n) converges to ℓ in E , then necessarily $\ell \in \overline{A}$.

Definition 1.57. Let $u: \mathbb{N} \rightarrow E$ be a sequence in a topological space E . A *subsequence* (or *extracted sequence*) of u is a sequence of the form $u \circ \varphi$, where $\varphi: \mathbb{N} \rightarrow \mathbb{N}$ is a strictly increasing function.

Proposition 1.58. If a sequence $(x_n)_{n \in \mathbb{N}}$ in a topological space E converges to a limit ℓ , then every subsequence of (x_n) also converges to ℓ .

Proof. Let $(x_{n_k})_{k \in \mathbb{N}}$ be a subsequence of (x_n) , so that the index map $k \mapsto n_k$ is strictly increasing. Let $V \in \mathcal{V}(\ell)$ be an arbitrary neighborhood of ℓ . Since $x_n \rightarrow \ell$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $x_n \in V$.

Because the sequence (n_k) is strictly increasing and unbounded, there exists $K \in \mathbb{N}$ such that for all $k \geq K$, $n_k \geq N$. It follows that for all $k \geq K$, $x_{n_k} \in V$. This proves that the subsequence $x_{n_k} \rightarrow \ell$. ■

1.9.1 Adherence Values of a Sequence

Definition 1.59. Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in a topological space E . A point $\ell \in E$ is an *adherence value* (or *cluster point*) of the sequence if

$$\forall V \in \mathcal{V}(\ell), \forall N \in \mathbb{N}, \exists n \geq N \text{ such that } x_n \in V.$$

In other words, every neighborhood of ℓ contains infinitely many terms of the sequence.

Every limit of a convergent sequence is an adherence value, but the converse is false in general.

Example 1.60. In \mathbb{R} , the sequence $x_n = (-1)^n$ has two adherence values: 1 and -1 .

Example 1.61. Consider the sequence defined by $u_0 = 0$, $u_1 = 1$, and $u_n = 2(-1)^n$ for $n \geq 2$. In the space $E = \{0, 1, 2, -2\}$, the adherence values of (u_n) are 2 and -2 .

Proposition 1.62. The set of adherence values of a sequence $u: \mathbb{N} \rightarrow E$ is equal to

$$\bigcap_{N \in \mathbb{N}} \overline{\{u_n \mid n \geq N\}}.$$

Proof. Let $\ell \in E$.

(\subseteq) Suppose ℓ is an adherence value of u . Let $N \in \mathbb{N}$ be arbitrary. We must show $\ell \in \overline{\{u_n \mid n \geq N\}}$. Let V be a neighborhood of ℓ . By definition of an adherence value, there exists some $n \geq N$ such that $u_n \in V$. This shows that every neighborhood of ℓ intersects the set $\{u_n \mid n \geq N\}$, so ℓ is in its closure. As this holds for all N , ℓ is in the intersection.

(\supseteq) Conversely, suppose $\ell \in \bigcap_{N \in \mathbb{N}} \overline{\{u_n \mid n \geq N\}}$. Let V be a neighborhood of ℓ and let $N \in \mathbb{N}$ be given. Since ℓ is in the closure of $\{u_n \mid n \geq N\}$, the neighborhood V must intersect this set. Therefore, there exists some $n \geq N$ with $u_n \in V$. This is precisely the condition for ℓ to be an adherence value of the sequence u . ■

Proposition 1.63. If a subsequence $(x_{\varphi(n)})_{n \in \mathbb{N}}$ of $(x_n)_{n \in \mathbb{N}}$ converges to ℓ , then ℓ is an adherence value of (x_n) .

Proof. Let V be a neighborhood of ℓ . Since the subsequence converges to ℓ , there exists N such that for all $n \geq N$, $x_{\varphi(n)} \in V$. As φ is strictly increasing, the indices $\varphi(n)$ are all distinct and arbitrarily large. Thus, V contains infinitely many terms of the original sequence, so ℓ is an adherence value. ■

Exercise 22. Let $(u_n)_{n \in \mathbb{N}}$ be a bounded sequence of real numbers. Prove that (u_n) converges if and only if it has a unique adherence value.

Solution 22. (\Rightarrow) If (u_n) converges, then by the previous proposition, its limit is its only adherence value.

(\Leftarrow) Suppose (u_n) has a unique adherence value ℓ but does not converge to ℓ . Then there exists $\varepsilon > 0$ such that for infinitely many n , $|u_n - \ell| \geq \varepsilon$. From these terms, we can extract a subsequence (u_{n_k}) . This subsequence is bounded, so by the Bolzano–Weierstrass theorem, it has a convergent subsequence, whose limit is an adherence value of (u_n) distinct from ℓ , a contradiction. Hence, (u_n) converges to ℓ .

Exercise 23. Determine the adherence values of the sequence $(u_n)_{n \in \mathbb{N}}$ defined by

$$u_n = (-1)^n + \frac{1}{n+1}.$$

Solution 23. The subsequence of even indices is $u_{2k} = 1 + \frac{1}{2k+1} \rightarrow 1$. The subsequence of odd indices is $u_{2k+1} = -1 + \frac{1}{2k+2} \rightarrow -1$.

Thus, 1 and -1 are adherence values. Any convergent subsequence must contain infinitely many even or infinitely many odd terms (or both), so its limit must be 1 or -1 . Therefore, the set of adherence values is $\{-1, 1\}$.

Exercise 24. Let $(u_n)_{n \in \mathbb{N}}$ be a bounded real sequence such that

$$u_n + \frac{1}{2}u_{2n} \xrightarrow{n \rightarrow \infty} 0.$$

1. Show that if a is an adherence value of (u_n) , then $-2a$ is also an adherence value.
2. Deduce that (u_n) converges.

Solution 24. 1. Let a be an adherence value. Then there is a subsequence $u_{n_k} \rightarrow a$. The corresponding subsequence u_{2n_k} is bounded, so it has a convergent subsequence $u_{2n_{k_j}} \rightarrow b$. From the given relation,

$$u_{n_{k_j}} + \frac{1}{2}u_{2n_{k_j}} \rightarrow 0,$$

we get $a + \frac{1}{2}b = 0$, so $b = -2a$. Thus, $-2a$ is an adherence value.

2. Suppose $a \neq 0$ is an adherence value. Then by (1), so are $-2a, 4a, -8a, \dots$, i.e., $(-2)^k a$ for all $k \in \mathbb{N}$. But this sequence is unbounded, contradicting the boundedness of (u_n) . Therefore, the only possible adherence value is 0. Since the sequence is bounded and has a unique adherence value, it converges to 0.

Exercise 25.

1. What are the adherence values of the sequences $(-1)^n$ and $\cos(n\pi/3)$?
2. Give an example of a non-convergent sequence that has a unique adherence value.

Solution 25. 1. The sequence $(-1)^n$ takes only the values 1 and -1 , and both are limits of subsequences (the even and odd terms), so its adherence values are $\{-1, 1\}$.

The sequence $\cos(n\pi/3)$ is periodic with period 6, taking the values $\cos(0) = 1$, $\cos(\pi/3) = 1/2$, $\cos(2\pi/3) = -1/2$, $\cos(\pi) = -1$, $\cos(4\pi/3) = -1/2$, $\cos(5\pi/3) = 1/2$. All these values are adherence values, so the set is $\{-1, -1/2, 1/2, 1\}$.

2. Define u_n by $u_{2n} = 1$ and $u_{2n+1} = n$. The sequence is divergent (since the odd terms go to infinity), but the only adherence value is 1, because any convergent subsequence must eventually consist only of even-indexed terms (since the odd terms are unbounded).

Exercise 26. Let $u = (u_n)_{n \in \mathbb{N}}$ be a real sequence such that $\lim_{n \rightarrow \infty} (u_{n+1} - u_n) = 0$. Prove that the set $\text{Adh}(u)$ of adherence values of u is an interval.

Solution 26. Let $a, b \in \text{Adh}(u)$ with $a < b$, and let $c \in]a, b[$. We will show $c \in \text{Adh}(u)$.

Let $\varepsilon > 0$. Since $u_{n+1} - u_n \rightarrow 0$, there exists N_0 such that $|u_{n+1} - u_n| < \varepsilon$ for all $n \geq N_0$.

Because a is an adherence value, there exists $n_1 \geq N_0$ with $u_{n_1} < c$. Because b is an adherence value, there exists $n_2 > n_1$ with $u_{n_2} > c$.

Consider the first index $p > n_1$ such that $u_p \geq c$. Then $u_{p-1} < c$ and $p - 1 \geq N_0$, so

$$c \leq u_p < u_{p-1} + \varepsilon < c + \varepsilon.$$

Also, $u_p \geq c > c - \varepsilon$. Thus, $u_p \in]c - \varepsilon, c + \varepsilon[$ and $p \geq N_0$.

Since ε and N_0 were arbitrary, c is an adherence value. Hence, $\text{Adh}(u)$ is an interval.

Exercise 27. Determine the set of adherence values, in \mathbb{R} , of the sequences u , v , and w defined by:

$$\forall n \in \mathbb{N}, u_{2n} = (-2)^n, u_{2n+1} = \sqrt{2}; \quad v_n = e^{-n}; \quad w_{2n} = 1, w_{2n+1} = n.$$

Solution 27. • For u_n : The odd subsequence is constant $\sqrt{2}$, so $\sqrt{2}$ is an adherence value. The even subsequence $(-2)^n$ is unbounded and divergent, and has no convergent subsequence. Thus, the only adherence value is $\{\sqrt{2}\}$.

- For $v_n = e^{-n}$: This sequence converges to 0, so its only adherence value is $\{0\}$.
- For w_n : The even subsequence is constant 1, so 1 is an adherence value. The odd subsequence n is unbounded, so it has no adherence values. Any convergent subsequence must be eventually even, so the only adherence value is $\{1\}$.

Exercise 28. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function, and let $u = (u_n)_{n \in \mathbb{N}}$ be a sequence defined by

$$u_0 \in \mathbb{R}, \quad u_{n+1} = f(u_n) \text{ for all } n \in \mathbb{N}.$$

Suppose that u has a unique adherence value a . Show that u converges to a .

Solution 28. Since f is continuous, if a subsequence $u_{n_k} \rightarrow a$, then $u_{n_k+1} = f(u_{n_k}) \rightarrow f(a)$. But u_{n_k+1} is also a subsequence of u , so its limit $f(a)$ must be an adherence value of u . By uniqueness, $f(a) = a$, so a is a fixed point.

Now, suppose for contradiction that u does not converge to a . Then there exists $\varepsilon > 0$ and a subsequence u_{m_k} such that $|u_{m_k} - a| \geq \varepsilon$ for all k . This subsequence is not necessarily bounded, but if it is, it has a convergent subsequence whose limit is an adherence value $\neq a$, a contradiction.

If it is unbounded, we can consider the set $K = \{x \in \mathbb{R} \mid |x - a| \geq \varepsilon\}$. The sequence visits K infinitely often. However, because a is the *only* adherence value, for any $M > 0$, the set $\{n \mid |u_n| > M\}$ must be finite (otherwise, $+\infty$ or $-\infty$ would be an adherence value in the extended reals, and we could argue the sequence must accumulate at a finite point or diverge, but the uniqueness of a precludes other finite accumulation points). A more direct argument: the set $\{u_n \mid n \in \mathbb{N}\}$ is relatively compact in \mathbb{R} because its only limit point is a . Thus, the sequence is bounded. By the Bolzano–Weierstrass theorem, the subsequence u_{m_k} has a convergent subsequence, whose limit is an adherence value different from a , a contradiction. Therefore, $u_n \rightarrow a$.

1.10 Continuous Functions

Throughout this section, (E, \mathcal{T}) and (E', \mathcal{T}') denote topological spaces.

1.10.1 Limits

Definition 1.64. Let $A \subseteq E$ be a nonempty subset, $f: A \rightarrow E'$ a function, $a \in \overline{A}$, and $b \in E'$. We say that $f(x)$ *tends to* b as x *tends to* a (and write $\lim_{x \rightarrow a} f(x) = b$) if

$$\forall W \in \mathcal{V}(b), \exists V \in \mathcal{V}(a) \text{ such that } f(V \cap A) \subseteq W.$$

Remark 1.65. In the above definition, the families $\mathcal{V}(b)$ and $\mathcal{V}(a)$ can be replaced by any neighborhood bases at b and a , respectively.

Definition 1.66 (Equivalent formulation). Let $f: E \rightarrow E'$ be a function, and let $a \in E$, $b \in E'$. Then $\lim_{x \rightarrow a} f(x) = b$ if and only if for every neighborhood W of b , the preimage $f^{-1}(W)$ is a neighborhood of a .

Theorem 1.67. If E' is a Hausdorff space, then the limit of a function at a point (if it exists) is unique.

Proof. Suppose f has two distinct limits $b \neq b'$ at a . Since E' is Hausdorff, there exist disjoint neighborhoods $W \in \mathcal{V}(b)$ and $W' \in \mathcal{V}(b')$. By the definition of limit, there exist $V \in \mathcal{V}(a)$ and $V' \in \mathcal{V}(a)$ such that $f(V \cap A) \subseteq W$ and $f(V' \cap A) \subseteq W'$. Then $f((V \cap V') \cap A) \subseteq W \cap W' = \emptyset$, so $(V \cap V') \cap A = \emptyset$. But this contradicts $a \in \overline{A}$, since $V \cap V'$ is a neighborhood of a . ■

Example 1.68. The function $f: \mathbb{R}_+^* \rightarrow \mathbb{R}$ defined by $f(x) = 1/x$ has no limit at 0 in \mathbb{R} . However, if we consider f as a function into the extended real line $\overline{\mathbb{R}}$, then $\lim_{x \rightarrow 0^+} f(x) = +\infty$.

1.10.2 Pointwise Continuity

Definition 1.69. A function $f: E \rightarrow E'$ is *continuous at a point* $a \in E$ if for every neighborhood W of $f(a)$, there exists a neighborhood V of a such that $f(V) \subseteq W$. Equivalently, $\lim_{x \rightarrow a} f(x) = f(a)$. The function f is *continuous (on E)* if it is continuous at every point of E .

Example 1.70. 1. Every constant map is continuous.

2. If E has the discrete topology, then every function $f: E \rightarrow E'$ is continuous.

3. The characteristic function $\mathbf{1}_{\mathbb{Q}}$ is discontinuous at every point of \mathbb{R} .

4. The function $\mathbf{1}_{\mathbb{R}_+}$ is continuous at every $x \neq 0$ but discontinuous at 0.

Proposition 1.71. A function $f: E \rightarrow E'$ is continuous at $a \in E$ if and only if the preimage under f of every neighborhood of $f(a)$ is a neighborhood of a .

Proposition 1.72. Let $f: E \rightarrow E'$ be a function. The following properties are equivalent:

1. f is continuous.
2. The preimage of every open set in E' is open in E .
3. The preimage of every closed set in E' is closed in E .
4. For every $A \subseteq E$, $f(\overline{A}) \subseteq \overline{f(A)}$.

5. For every $B \subseteq E'$, $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$.

Proof. The equivalence of (2) and (3) follows by taking complements and using the identity $f^{-1}(E' \setminus B) = E \setminus f^{-1}(B)$.

(1) \iff (2): f is continuous iff for every $a \in E$ and every open neighborhood O of $f(a)$, $f^{-1}(O)$ is a neighborhood of a . This is equivalent to saying that $f^{-1}(O)$ is a neighborhood of each of its points, i.e., $f^{-1}(O)$ is open.

(3) \implies (5): If $F = \overline{B}$ is closed, then $f^{-1}(F)$ is closed and contains $f^{-1}(B)$, so it contains $\overline{f^{-1}(B)}$.

(5) \implies (4): Let $B = f(A)$. Then $A \subseteq f^{-1}(B)$, so $\overline{A} \subseteq \overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$, which implies $f(\overline{A}) \subseteq \overline{B} = f(A)$.

(4) \implies (3): Let $F \subseteq E'$ be closed and set $A = f^{-1}(F)$. Then $f(A) \subseteq F$, so $\overline{f(A)} \subseteq F$ (since F is closed). By (4), $f(\overline{A}) \subseteq \overline{f(A)} \subseteq F$, so $\overline{A} \subseteq f^{-1}(F) = A$. Thus, A is closed. ■

Example 1.73. 1. If $f: E \rightarrow \mathbb{R}$ is continuous, then for any $\alpha, \beta \in \mathbb{R}$:

$$\{x \in E \mid \alpha \leq f(x) \leq \beta\} = f^{-1}([\alpha, \beta]) \text{ is closed,}$$

$$\{x \in E \mid f(x) > \alpha\} = f^{-1}(] \alpha, +\infty[) \text{ is open,}$$

$$\{x \in E \mid f(x) = \alpha\} = f^{-1}(\{\alpha\}) \text{ is closed.}$$

2. For any $a \in \mathbb{R}$, the translation map $t_a: \mathbb{R} \rightarrow \mathbb{R}$, $t_a(x) = x + a$, is continuous. Indeed, the preimage of an open interval $] \alpha, \beta [$ is $] \alpha - a, \beta - a [$, which is open.

3. The characteristic function $\mathbf{1}_{\mathbb{R}_+}$ is not continuous: the preimage of the open set $]1/2, 3/2[$ is \mathbb{R}_+ , which is not open in \mathbb{R} .

Remark 1.74. The direct image of an open (resp. closed) set under a continuous function is not necessarily open (resp. closed). For example, $\sin(] - 10, 13[) = [-1, 1]$.

Definition 1.75. A function is called *open* (resp. *closed*) if the image of every open (resp. closed) set is open (resp. closed).

Example 1.76. Let E_1 and E_2 be topological spaces, and let $E_1 \times E_2$ have the product topology. The projection $\pi_1: E_1 \times E_2 \rightarrow E_1$, $\pi_1(x_1, x_2) = x_1$, is continuous and open. It is continuous because $\pi_1^{-1}(O) = O \times E_2$ is open for any open $O \subseteq E_1$. It is open because the image of an elementary open set $O_1 \times O_2$ is O_1 , and arbitrary unions of open sets are open.

1.10.3 Composition of Continuous Functions

Proposition 1.77. Let $f: E \rightarrow E'$ and $g: E' \rightarrow E''$ be functions between topological spaces.

1. If f and g are continuous, then $g \circ f$ is continuous.
2. If f is continuous at $x \in E$ and g is continuous at $f(x) \in E'$, then $g \circ f$ is continuous at x .

Proof. (1) If $O \subseteq E''$ is open, then $(g \circ f)^{-1}(O) = f^{-1}(g^{-1}(O))$ is open in E by the continuity of g and f .

(2) Let V be a neighborhood of $(g \circ f)(x) = g(f(x))$. Since g is continuous at $f(x)$, $g^{-1}(V)$ is a neighborhood of $f(x)$. Since f is continuous at x , $f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V)$ is a neighborhood of x . ■

1.11 Homeomorphisms

Homeomorphisms are the isomorphisms of topological spaces; they allow us to identify two topologically equivalent spaces.

Definition 1.78. A *homeomorphism* between two topological spaces (E, \mathcal{T}_E) and $(E', \mathcal{T}_{E'})$ is a bijective continuous function $f: E \rightarrow E'$ whose inverse f^{-1} is also continuous. Two spaces are said to be *homeomorphic* if there exists a homeomorphism between them.

Remark 1.79. • The relation "is homeomorphic to" is an equivalence relation.

- The composition of two homeomorphisms is a homeomorphism.

Example 1.80. 1. Translations and homotheties on \mathbb{R} are homeomorphisms. It follows that any two open intervals in \mathbb{R} are homeomorphic. In fact, every open interval is homeomorphic to \mathbb{R} itself.

2. The identity map $\text{Id}: (E, \mathcal{T}) \rightarrow (E, \mathcal{T}')$ is a homeomorphism if and only if $\mathcal{T} = \mathcal{T}'$.

3. A continuous bijection is not necessarily a homeomorphism. For example, the map $f: [0, 1[\cup \{2\} \rightarrow [0, 1]$ defined by $f|_{[0, 1[} = \text{Id}$ and $f(2) = 1$ is a continuous bijection, but its inverse is not continuous at 1.

Definition 1.81. A property \mathcal{P} of a topological space is called a *topological property* if it is preserved under homeomorphisms; that is, if a space E has property \mathcal{P} , then any space homeomorphic to E also has property \mathcal{P} .

Example 1.82. The following are topological properties: being open, closed, or a neighborhood of a point; being Hausdorff; being the closure, interior, or boundary of a set.

1.12 Topology of Metric Spaces

Let E be a non-empty set.

Definition 1.83. A *metric* (or *distance*) on E is a function $d: E \times E \rightarrow \mathbb{R}_+$ satisfying the following properties for all $x, y, z \in E$:

1. $d(x, x) = 0$.
2. $d(x, y) = d(y, x)$ (symmetry).
3. $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality).
4. $d(x, y) = 0 \implies x = y$ (separation axiom).

The number $d(x, y)$ is called the *distance* between x and y . A set E equipped with a metric d is called a *metric space* and is denoted by (E, d) .

If a function d satisfies properties (1), (2), and (3) but not necessarily (4), it is called a *pseudo-metric* (or *semi-metric*), and the pair (E, d) is a *pseudo-metric space*.

Example 1.84. 1. **Standard metric on \mathbb{R} :** $d(x, y) = |x - y|$.

2. **Standard metric on \mathbb{C} :** $d(z_1, z_2) = |z_1 - z_2|$.

3. Discrete metric on any set E :

$$d(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y. \end{cases}$$

Indeed, this function satisfies all the axioms of a metric:

- $d(x, x) = 0$ for all $x \in E$.
- $d(x, y) = d(y, x)$ by symmetry of the definition.
- The triangle inequality holds: for any $x, y, z \in E$, if $x = z$ then $d(x, z) = 0 \leq d(x, y) + d(y, z)$. If $x \neq z$, then either $x \neq y$ or $y \neq z$ (or both), so $d(x, y) + d(y, z) \geq 1 = d(x, z)$.
- $d(x, y) = 0$ implies $x = y$ by definition.

Thus, (E, d) is a metric space.

4. Euclidean metric on \mathbb{R}^n :

$$d_e(x, y) = \sqrt{\sum_{i=1}^n |x_i - y_i|^2}, \quad \text{where } x = (x_1, \dots, x_n), y = (y_1, \dots, y_n).$$

5. Hölder metric of exponent p on \mathbb{R}^n (for $p \geq 1$):

$$d_p(x, y) = \left(\sum_{i=1}^n |x_i - y_i|^p \right)^{1/p}.$$

The triangle inequality follows from Minkowski's inequality. As $p \rightarrow \infty$, this converges to the *supremum metric*:

$$d_\infty(x, y) = \max_{1 \leq i \leq n} |x_i - y_i|.$$

6. Uniform convergence metric on $\mathcal{F}(A, E)$: Let A be a set and (E, d) a metric space. On the set $\mathcal{F}(A, E)$ of all functions from A to E , the function

$$d_u(f, g) = \sup_{x \in A} \min\{1, d(f(x), g(x))\}$$

defines a metric, called the *uniform convergence metric*.

7. Metrics on $C([a, b], \mathbb{C})$: On the space of continuous complex-valued functions on $[a, b]$, the following are metrics:

$$d_1(f, g) = \int_a^b |f(x) - g(x)| dx, \quad d_2(f, g) = \left(\int_a^b |f(x) - g(x)|^2 dx \right)^{1/2}.$$

We verify the metric axioms for both d_1 and d_2 .

1. The distance d_1 :

- *Non-negativity*: Since $|f(x) - g(x)| \geq 0$ for all x , the integral is non-negative: $d_1(f, g) \geq 0$.
- *Separation*: If $f = g$, then $d_1(f, g) = 0$. Conversely, if $d_1(f, g) = 0$, then $\int_a^b |f(x) - g(x)| dx = 0$. Since the integrand is a continuous non-negative function, this implies $|f(x) - g(x)| = 0$ for all $x \in [a, b]$, so $f = g$.

- *Symmetry*: $|f(x) - g(x)| = |g(x) - f(x)|$, so $d_1(f, g) = d_1(g, f)$.
- *Triangle inequality*: For any $f, g, h \in E$, we have the pointwise inequality

$$|f(x) - h(x)| \leq |f(x) - g(x)| + |g(x) - h(x)|.$$

Integrating both sides over $[a, b]$ yields

$$d_1(f, h) \leq d_1(f, g) + d_1(g, h).$$

Thus, d_1 is a metric on E .

2. The distance d_2 :

- *Non-negativity and symmetry* are immediate, as in the case of d_1 .
- *Separation*: If $d_2(f, g) = 0$, then $\int_a^b |f(x) - g(x)|^2 dx = 0$. The integrand is a continuous non-negative function, so it must be identically zero, implying $f = g$.
- *Triangle inequality*: This follows from the *Minkowski inequality* for $p = 2$:

$$\left(\int_a^b |f(x) - h(x)|^2 dx \right)^{1/2} \leq \left(\int_a^b |f(x) - g(x)|^2 dx \right)^{1/2} + \left(\int_a^b |g(x) - h(x)|^2 dx \right)^{1/2}.$$

A direct proof uses the Cauchy–Schwarz inequality. Let $u = f - g$ and $v = g - h$. Then

$$\|u + v\|_2^2 = \|u\|_2^2 + 2 \operatorname{Re}\langle u, v \rangle + \|v\|_2^2 \leq \|u\|_2^2 + 2\|u\|_2\|v\|_2 + \|v\|_2^2 = (\|u\|_2 + \|v\|_2)^2,$$

where $\langle u, v \rangle = \int_a^b u(x)\overline{v(x)} dx$ is the standard inner product on $C([a, b], \mathbb{C})$. Taking square roots gives the triangle inequality.

Therefore, d_2 is also a metric on E .

Proposition 1.85. Let E be a non-empty set equipped with a metric d , and let n be a positive integer.

1. (*Generalized triangle inequality*) For all $(x_1, \dots, x_n) \in E^n$,

$$d(x_1, x_n) \leq \sum_{i=1}^{n-1} d(x_i, x_{i+1}).$$

2. (*Reverse triangle inequality*) For all $(x, y, z) \in E^3$,

$$|d(x, z) - d(y, z)| \leq d(x, y).$$

3. For every $\lambda \in \mathbb{R}_+^*$, the function λd is also a metric on E .

Proof. 1. The proof is by induction on n . The case $n = 2$ is trivial, and $n = 3$ is the standard triangle inequality. Assume the property holds for some $n \geq 3$. For a sequence (x_1, \dots, x_{n+1}) , we have

$$d(x_1, x_{n+1}) \leq d(x_1, x_n) + d(x_n, x_{n+1}) \leq \sum_{i=1}^{n-1} d(x_i, x_{i+1}) + d(x_n, x_{n+1}) = \sum_{i=1}^n d(x_i, x_{i+1}),$$

which completes the induction.

2. By the triangle inequality, $d(x, z) \leq d(x, y) + d(y, z)$, so $d(x, z) - d(y, z) \leq d(x, y)$. Similarly, $d(y, z) \leq d(y, x) + d(x, z) = d(x, y) + d(x, z)$, so $d(y, z) - d(x, z) \leq d(x, y)$, which is equivalent to $-(d(x, z) - d(y, z)) \leq d(x, y)$. Combining these two inequalities yields $|d(x, z) - d(y, z)| \leq d(x, y)$.

3. Let $\lambda > 0$. We verify the metric axioms for $d'(x, y) = \lambda d(x, y)$.

- $d'(x, y) = 0 \iff \lambda d(x, y) = 0 \iff d(x, y) = 0 \iff x = y$.
- $d'(x, y) = \lambda d(x, y) = \lambda d(y, x) = d'(y, x)$.
- $d'(x, z) = \lambda d(x, z) \leq \lambda(d(x, y) + d(y, z)) = d'(x, y) + d'(y, z)$.

Thus, λd is a metric. ■

Exercise 29.

1. Under what condition on a function $f: \mathbb{R} \rightarrow \mathbb{R}$ does the mapping $d(x, y) = |f(x) - f(y)|$ define a metric on \mathbb{R} ?

2. Determine whether the following mappings define metrics on \mathbb{R} :

$$\delta_1(x, y) = |\sin x - \sin y|, \quad \delta_2(x, y) = |x^2 - y^2|, \quad \delta_3(x, y) = |x^3 - y^3|, \quad \delta_4(x, y) = \log(1 + |x - y|).$$

3. Prove that the mapping

$$d(x, y) = \left| \frac{x}{1 + |x|} - \frac{y}{1 + |y|} \right|$$

defines a metric on \mathbb{R} .

Solution 29. 1. The mapping $d(x, y) = |f(x) - f(y)|$ is a metric if and only if f is *injective*. The metric axioms of non-negativity, symmetry, and the triangle inequality are automatic. The separation axiom $d(x, y) = 0 \iff x = y$ is equivalent to $f(x) = f(y) \iff x = y$, which is injectivity.

- 2.
- δ_1 : Not a metric, since $f(x) = \sin x$ is not injective ($\sin 0 = \sin \pi$).
 - δ_2 : Not a metric, since $f(x) = x^2$ is not injective ($1^2 = (-1)^2$).
 - δ_3 : Is a metric, since $f(x) = x^3$ is injective.
 - δ_4 : Is a metric. The function $\varphi(t) = \log(1 + t)$ is increasing, $\varphi(0) = 0$, and is subadditive for $t \geq 0$, so $d = \varphi \circ d_{\text{std}}$ is a metric.

3. The function $f(x) = x/(1 + |x|)$ is a continuous, strictly increasing bijection from \mathbb{R} to $(-1, 1)$. Since it is injective, the mapping $d(x, y) = |f(x) - f(y)|$ is a metric.

Exercise 30. Prove that the mapping $d: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}_+$ defined by

$$d(x, y) = \frac{|x - y|}{1 + |x - y|}$$

is a metric on \mathbb{R} .

Solution 30. We verify the four axioms of a metric.

- *Non-negativity*: Since $|x - y| \geq 0$, it follows that $d(x, y) \geq 0$ for all $x, y \in \mathbb{R}$.

- *Separation:* $d(x, y) = 0$ if and only if $\frac{|x-y|}{1+|x-y|} = 0$, which is equivalent to $|x - y| = 0$, i.e., $x = y$.
- *Symmetry:* $d(x, y) = \frac{|x-y|}{1+|x-y|} = \frac{|y-x|}{1+|y-x|} = d(y, x)$.
- *Triangle inequality:* This is the non-trivial part. Let $x, y, z \in \mathbb{R}$. Define the function $f: [0, \infty) \rightarrow [0, 1)$ by

$$f(t) = \frac{t}{1+t}.$$

The function f is **increasing** (since $f'(t) = \frac{1}{(1+t)^2} > 0$) and **subadditive**, meaning $f(a+b) \leq f(a) + f(b)$ for all $a, b \geq 0$. To see this, note that

$$f(a+b) = \frac{a+b}{1+a+b} = \frac{a}{1+a+b} + \frac{b}{1+a+b} \leq \frac{a}{1+a} + \frac{b}{1+b} = f(a) + f(b),$$

since $1+a+b \geq 1+a$ and $1+a+b \geq 1+b$.

Now, by the standard triangle inequality for the absolute value, we have

$$|x-z| \leq |x-y| + |y-z|.$$

Since f is increasing,

$$d(x, z) = f(|x-z|) \leq f(|x-y| + |y-z|).$$

By the subadditivity of f ,

$$f(|x-y| + |y-z|) \leq f(|x-y|) + f(|y-z|) = d(x, y) + d(y, z).$$

Combining these, we obtain $d(x, z) \leq d(x, y) + d(y, z)$.

Therefore, d is a metric on \mathbb{R} .

Exercise 31. Let (E, d) be a metric space. For $\alpha > 0$, define the mapping $d^\alpha: E \times E \rightarrow \mathbb{R}$ by $d^\alpha(x, y) = (d(x, y))^\alpha$.

1. Show that for any $\alpha \in (0, 1]$, the mapping d^α defines a metric on E .
2. Show that there exists a number $\alpha^* = \alpha^*(d) \in [1, \infty]$ such that:
 - If $0 < \alpha < \alpha^*$, then d^α is a metric on E .
 - If $\alpha > \alpha^*$, then d^α is not a metric on E .
3. Determine α^* in the following cases:
 - (a) d is the standard metric on \mathbb{R} .
 - (b) d is the discrete metric on an arbitrary set E .

Solution 31. 1. Let $\varphi(t) = t^\alpha$ for $t \geq 0$. We must show that φ is *subadditive*, i.e., $\varphi(a+b) \leq \varphi(a) + \varphi(b)$ for all $a, b \geq 0$.

If $a = 0$ or $b = 0$, the inequality is trivial. Assume $a, b > 0$ and set $t = b/a \geq 0$. The inequality becomes $(1+t)^\alpha \leq 1 + t^\alpha$. Define $f(t) = (1+t)^\alpha - (1+t^\alpha)$ for $t \geq 0$. Then $f(0) = 0$, and

$$f'(t) = \alpha \left((1+t)^{\alpha-1} - t^{\alpha-1} \right).$$

Since $\alpha - 1 \leq 0$ and $t \leq 1 + t$, we have $(1 + t)^{\alpha-1} \leq t^{\alpha-1}$, so $f'(t) \leq 0$. Thus, f is non-increasing, and since $f(0) = 0$, we have $f(t) \leq 0$ for all $t \geq 0$.

This proves that φ is subadditive. Therefore, for any $x, y, z \in E$,

$$d^\alpha(x, z) = \varphi(d(x, z)) \leq \varphi(d(x, y) + d(y, z)) \leq \varphi(d(x, y)) + \varphi(d(y, z)) = d^\alpha(x, y) + d^\alpha(y, z),$$

so the triangle inequality holds. The other metric axioms are immediate, so d^α is a metric for all $\alpha \in (0, 1]$.

2. Define the set

$$I = \{\alpha > 0 \mid d^\alpha \text{ is a metric on } E\}.$$

From part 1, $(0, 1] \subseteq I$, so I is nonempty. We now show that I is an interval. Suppose $\alpha \in I$ and $0 < \beta < \alpha$. Then

$$d^\beta = (d^\alpha)^{\beta/\alpha}.$$

Since $\beta/\alpha \in (0, 1]$ and d^α is a metric, part 1 implies that $(d^\alpha)^{\beta/\alpha}$ is also a metric. Hence $\beta \in I$, so I is an interval.

Let $\alpha^* = \sup I$. By the definition of the supremum, if $0 < \alpha < \alpha^*$, then $\alpha \in I$, so d^α is a metric. If $\alpha > \alpha^*$, then $\alpha \notin I$, so d^α is not a metric. This establishes the existence of the desired $\alpha^* \in [1, \infty]$.

3. We now determine α^* for the two specific cases.

[label=(c), leftmargin=*]

(a) Let d be the standard metric on \mathbb{R} , so $d(x, y) = |x - y|$. We claim $\alpha^* = 1$.

From part 1, d^α is a metric for all $\alpha \in (0, 1]$, so $\alpha^* \geq 1$. Now let $\alpha > 1$. Consider the points $x = 0$, $y = 1$, $z = 2$. Then

$$d^\alpha(x, z) = 2^\alpha > 2 = 1^\alpha + 1^\alpha = d^\alpha(x, y) + d^\alpha(y, z),$$

so the triangle inequality fails. Thus, $\alpha \notin I$ for all $\alpha > 1$, and therefore $\alpha^* = 1$.

(b) Let d be the discrete metric on E , defined by $d(x, y) = 0$ if $x = y$ and $d(x, y) = 1$ if $x \neq y$.

For any $\alpha > 0$, if $x = y$, then $d^\alpha(x, y) = 0^\alpha = 0$. If $x \neq y$, then $d^\alpha(x, y) = 1^\alpha = 1$. Hence $d^\alpha = d$ for all $\alpha > 0$, so $I = (0, \infty)$ and $\alpha^* = \sup I = \infty$.

Exercise 32. Let (E, d) be a metric space, and let $\varphi: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be an increasing function such that $\varphi(t) = 0 \iff t = 0$ and $\varphi(a + b) \leq \varphi(a) + \varphi(b)$ for all $a, b \geq 0$.

1. Show that $d' = \varphi \circ d$ is a metric on E .

2. Show that $d_1 = \frac{d}{1 + d}$ and $d_2 = \log(1 + d)$ are metrics on E .

Solution 32. 1. We verify the four axioms of a metric for $d'(x, y) = \varphi(d(x, y))$.

- *Non-negativity:* Since φ maps to \mathbb{R}_+ , $d'(x, y) \geq 0$ for all $x, y \in E$.
- *Separation:* $d'(x, y) = 0 \iff \varphi(d(x, y)) = 0 \iff d(x, y) = 0 \iff x = y$.
- *Symmetry:* $d'(x, y) = \varphi(d(x, y)) = \varphi(d(y, x)) = d'(y, x)$.

- *Triangle inequality:* Since d is a metric, $d(x, z) \leq d(x, y) + d(y, z)$. As φ is increasing,

$$\varphi(d(x, z)) \leq \varphi(d(x, y) + d(y, z)).$$

By the subadditivity of φ ,

$$\varphi(d(x, y) + d(y, z)) \leq \varphi(d(x, y)) + \varphi(d(y, z)).$$

Combining these, we get $d'(x, z) \leq d'(x, y) + d'(y, z)$.

Thus, d' is a metric on E .

2. We apply the result from part 1 by showing that the corresponding φ functions satisfy the required properties.

- For $d_1 = \frac{d}{1+d}$, define $\varphi_1(t) = \frac{t}{1+t}$ for $t \geq 0$.

$$- \varphi_1(t) = 0 \iff t = 0.$$

$$- \varphi_1 \text{ is increasing on } \mathbb{R}_+ \text{ (its derivative } \varphi_1'(t) = \frac{1}{(1+t)^2} > 0).$$

$$- \varphi_1 \text{ is subadditive: for } a, b \geq 0,$$

$$\varphi_1(a+b) = \frac{a+b}{1+a+b} = \frac{a}{1+a+b} + \frac{b}{1+a+b} \leq \frac{a}{1+a} + \frac{b}{1+b} = \varphi_1(a) + \varphi_1(b).$$

Therefore, $d_1 = \varphi_1 \circ d$ is a metric.

- For $d_2 = \log(1+d)$, define $\varphi_2(t) = \log(1+t)$ for $t \geq 0$.

$$- \varphi_2(t) = 0 \iff t = 0.$$

$$- \varphi_2 \text{ is increasing on } \mathbb{R}_+ \text{ (its derivative } \varphi_2'(t) = \frac{1}{1+t} > 0).$$

$$- \varphi_2 \text{ is subadditive: for } a, b \geq 0,$$

$$\varphi_2(a+b) = \log(1+a+b) \leq \log(1+a+b+ab) = \log((1+a)(1+b)) = \log(1+a) + \log(1+b),$$

which implies that

$$\varphi_2(a+b) \leq \varphi_2(a) + \varphi_2(b).$$

Therefore, $d_2 = \varphi_2 \circ d$ is a metric.

1.12.1 Balls and Spheres

Definition 1.86. Let (E, d) be a metric space, let $x \in E$, and let $r > 0$.

- The set

$$B(x, r) = \{y \in E \mid d(x, y) < r\}$$

is called the *open ball* with center x and radius r .

- The set

$$\bar{B}(x, r) = \{y \in E \mid d(x, y) \leq r\}$$

is called the *closed ball* with center x and radius r .

- The set

$$S(x, r) = \{y \in E \mid d(x, y) = r\}$$

is called the *sphere* with center x and radius r .

Example 1.87. 1. **Standard metric on \mathbb{R} .** For the standard metric $d(x, y) = |x - y|$, the open ball is

$$B(x, r) = \{y \in \mathbb{R} \mid |x - y| < r\} =]x - r, x + r[.$$

Conversely, every bounded open interval $]a, b[$ can be written uniquely as an open ball:

$$]a, b[= B\left(\frac{a + b}{2}, \frac{b - a}{2}\right).$$

Thus, the open balls in \mathbb{R} are precisely the bounded open intervals. Similarly, the closed balls are the bounded closed intervals $[a, b]$.

2. **Metric $d(x, y) = |\arctan x - \arctan y|$ on \mathbb{R} .** The function $\arctan: \mathbb{R} \rightarrow]-\pi/2, \pi/2[$ is a homeomorphism. The open balls in this metric are the preimages under \arctan of open intervals in $]-\pi/2, \pi/2[$. They are:

$$]a, b[, \quad]a, +\infty[, \quad]-\infty, b[, \quad \text{and } \mathbb{R},$$

for $a, b \in \mathbb{R}$. Note that the unbounded intervals (e.g., $]a, +\infty[$) can be expressed as open balls, but the center and radius are not unique.

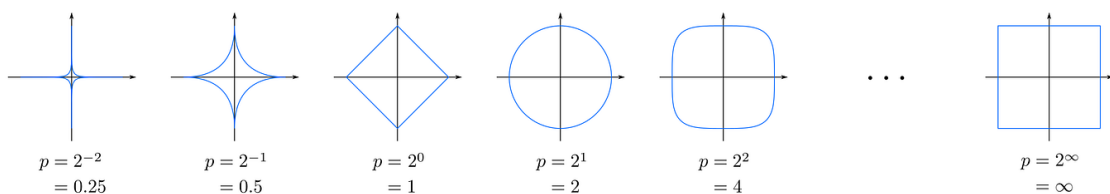
3. **Minkowski metrics on \mathbb{R}^n .** For $p \geq 1$, the Minkowski distance of order p between $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ is

$$d_p(x, y) = \left(\sum_{i=1}^n |x_i - y_i|^p\right)^{1/p}.$$

Important special cases include:

- *Manhattan distance* ($p = 1$): $d_1(x, y) = \sum_{i=1}^n |x_i - y_i|$.
- *Euclidean distance* ($p = 2$): $d_2(x, y) = \sqrt{\sum_{i=1}^n |x_i - y_i|^2}$.
- *Chebyshev distance* ($p \rightarrow \infty$): $d_\infty(x, y) = \max_{1 \leq i \leq n} |x_i - y_i|$.

The shapes of the open balls $B(0, 1)$ in these metrics are very different: a diamond for $p = 1$, a circle for $p = 2$, and a square for $p = \infty$.



1.12.2 Characterization of Neighborhoods and Open Sets in a Metric Space

Definition 1.88. Let (E, d) be a metric space. A subset $O \subseteq E$ is said to be *open* if for every $x \in O$, there exists a radius $r_x > 0$ such that the open ball $B(x, r_x) \subseteq O$.

Proposition 1.89. Let (E, d) be a metric space.

1. Every open ball in E is an open set.
2. Every closed ball in E is a closed set.

Proof. 1. Let $B(x, r)$ be an open ball, and let $y \in B(x, r)$. Then $d(x, y) < r$. Choose a radius $r_y > 0$ such that $r_y < r - d(x, y)$. For any $z \in B(y, r_y)$, the triangle inequality gives

$$d(z, x) \leq d(z, y) + d(y, x) < r_y + d(x, y) < (r - d(x, y)) + d(x, y) = r.$$

Therefore, $z \in B(x, r)$, so $B(y, r_y) \subseteq B(x, r)$. This shows that $B(x, r)$ is open.

2. Let $\overline{B}(x, r)$ be a closed ball. We will show its complement is open. Let $y \in E \setminus \overline{B}(x, r)$, so $d(x, y) > r$. Define $\delta = d(x, y) - r > 0$. For any $z \in B(y, \delta)$, the reverse triangle inequality gives

$$d(z, x) \geq d(y, x) - d(z, y) > d(x, y) - \delta = r.$$

Hence, $z \in E \setminus \overline{B}(x, r)$, so $B(y, \delta) \subseteq E \setminus \overline{B}(x, r)$. Thus, the complement is open, and $\overline{B}(x, r)$ is closed. ■

Proposition 1.90. Let (E, d) be a metric space, let $x \in E$, and let $V \subseteq E$. Then V is a neighborhood of x if and only if there exists $r > 0$ such that $B(x, r) \subseteq V$.

Proof. (\Rightarrow) If V is a neighborhood of x , there exists an open set O such that $x \in O \subseteq V$. By the definition of an open set in a metric space, there exists $r > 0$ such that $B(x, r) \subseteq O \subseteq V$.

(\Leftarrow) Conversely, if $B(x, r) \subseteq V$ for some $r > 0$, then since $B(x, r)$ is an open set containing x , its superset V is a neighborhood of x by definition. ■

Proposition 1.91. Let (E, d) be a metric space.

1. For every $x \in E$, the countable family $\{B(x, 1/n) \mid n \in \mathbb{N}^*\}$ is a neighborhood basis at x .
2. The family $\mathcal{B} = \{B(x, 1/n) \mid x \in E, n \in \mathbb{N}^*\}$ is a basis for the topology of (E, d) .

Proof. 1. Let V be an arbitrary neighborhood of x . Then there exists $r > 0$ such that $B(x, r) \subseteq V$. By the Archimedean property, there exists $n \in \mathbb{N}^*$ such that $1/n < r$. It follows that $B(x, 1/n) \subseteq B(x, r) \subseteq V$, which proves the claim.

2. To show \mathcal{B} is a basis, we must verify that every open set in E is a union of elements of \mathcal{B} . Let $O \subseteq E$ be open and let $x \in O$. There exists $r > 0$ such that $B(x, r) \subseteq O$. Choose n such that $1/n < r$. Then $x \in B(x, 1/n) \subseteq B(x, r) \subseteq O$. Thus, $O = \bigcup_{x \in O} B(x, 1/n_x)$ for suitable n_x , which is a union of elements of \mathcal{B} . ■

1.12.3 Distance from a Point to a Set, from a Set to Another, and Diameter

Definition 1.92. Let (E, d) be a metric space, and let $A, B \subseteq E$ be non-empty subsets. For a point $x \in E$, we define:

1. The *distance between the sets* A and B is

$$d(A, B) = \inf\{d(a, b) \mid a \in A, b \in B\}.$$

2. The *distance from the point* x *to the set* A is

$$d(x, A) = \inf\{d(x, a) \mid a \in A\}.$$

3. The *diameter* of A is the element of $\mathbb{R}_+ \cup \{+\infty\}$ defined by

$$\delta(A) = \sup\{d(x, y) \mid x, y \in A\}.$$

The set A is said to be *bounded* if $\delta(A) < +\infty$. Equivalently, A is bounded if there exists a point $a \in E$ and a radius $r > 0$ such that $A \subseteq B(a, r)$.

Remark 1.93. 1. The "distance" between sets is not a metric on the power set of E . For example, in \mathbb{R} with the standard metric, let $A = \{2\}$ and $B = \{2 + 1/(n + 1) \mid n \in \mathbb{N}\}$. Then $d(A, B) = 0$ even though $A \neq B$.

2. The diameter of an open ball is at most twice its radius: $\delta(B(a, r)) \leq 2r$.

3. The distance between sets can be expressed as

$$d(A, B) = \inf_{x \in A} d(x, B) = \inf_{y \in B} d(y, A).$$

Proposition 1.94. Let A and B be subsets of a metric space (E, d) . Then:

1. If $A \subseteq B$, then $\delta(A) \leq \delta(B)$.
2. $\delta(A) = \delta(\bar{A})$, where \bar{A} is the closure of A .

Proof. 1. If $A \subseteq B$, then every pair of points in A is also a pair of points in B . Therefore, the set of distances $\{d(x, y) \mid x, y \in A\}$ is a subset of $\{d(x, y) \mid x, y \in B\}$. The supremum of a subset is less than or equal to the supremum of the whole set, so $\delta(A) \leq \delta(B)$.

2. Since $A \subseteq \bar{A}$, we have $\delta(A) \leq \delta(\bar{A})$ by part 1. For the reverse inequality, let $x, y \in \bar{A}$. By definition of the closure, there exist sequences $(x_n), (y_n) \subseteq A$ such that $x_n \rightarrow x$ and $y_n \rightarrow y$. The metric d is continuous, so $d(x_n, y_n) \rightarrow d(x, y)$. Since $d(x_n, y_n) \leq \delta(A)$ for all n , it follows that $d(x, y) \leq \delta(A)$. As this holds for all $x, y \in \bar{A}$, we have $\delta(\bar{A}) \leq \delta(A)$. ■

Definition 1.95. Let Λ be a non-empty set and (E, d) a metric space. A function $f: \Lambda \rightarrow E$ is said to be *bounded* if its image $f(\Lambda)$ is a bounded subset of E .

Exercise 33. Let (E, d) be the metric space \mathbb{R}^2 with the Euclidean distance

$$d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

Let $\Lambda = \{1, 2, 3, 4\}$ and define $f: \Lambda \rightarrow E$ by

$$f(1) = (2, 3), \quad f(2) = (5, 7), \quad f(3) = (-1, -2), \quad f(4) = (0, 0).$$

1. Verify that the set $f(\Lambda) = \{(2, 3), (5, 7), (-1, -2), (0, 0)\}$ is bounded in \mathbb{R}^2 .
2. Calculate the maximum distance between any two points in $f(\Lambda)$.
3. Find a bound M such that for all $x, y \in f(\Lambda)$, $d(x, y) \leq M$.

Solution 33. 1. A finite subset of a metric space is always bounded, since the maximum of a finite number of real numbers is finite.

2. We compute the Euclidean distance for each pair of distinct points:

$$\begin{aligned} d(f(1), f(2)) &= \sqrt{(5-2)^2 + (7-3)^2} = \sqrt{9+16} = \sqrt{25} = 5, \\ d(f(1), f(3)) &= \sqrt{(-1-2)^2 + (-2-3)^2} = \sqrt{9+25} = \sqrt{34} \approx 5.83, \\ d(f(1), f(4)) &= \sqrt{(0-2)^2 + (0-3)^2} = \sqrt{4+9} = \sqrt{13} \approx 3.61, \\ d(f(2), f(3)) &= \sqrt{(-1-5)^2 + (-2-7)^2} = \sqrt{36+81} = \sqrt{117} \approx 10.82, \\ d(f(2), f(4)) &= \sqrt{(0-5)^2 + (0-7)^2} = \sqrt{25+49} = \sqrt{74} \approx 8.60, \\ d(f(3), f(4)) &= \sqrt{(0+1)^2 + (0+2)^2} = \sqrt{1+4} = \sqrt{5} \approx 2.24. \end{aligned}$$

The maximum distance is $\sqrt{117}$, between the points $(5, 7)$ and $(-1, -2)$.

3. We can take $M = \sqrt{117}$ as a bound. Thus, for all $x, y \in f(\Lambda)$, $d(x, y) \leq \sqrt{117}$.

Exercise 34. Metrics on \mathbb{R}_+^* Let $E = \mathbb{R}_+^* = (0, \infty)$ and define $d: E \times E \rightarrow \mathbb{R}_+$ by

$$d(x, y) = \left| \frac{1}{x} - \frac{1}{y} \right|.$$

1. Verify that d is a metric on E .
2. Compute the open balls $B_d(1, 1)$ and $B_d\left(\frac{1}{2}, 3\right)$.
3. Determine a general description of all open balls for this metric.
4. Determine whether the segment $(0, 1]$ (viewed as a subset of E) is bounded with respect to this metric. Is it closed?
5. Give an example of a set that is bounded for the metric d but unbounded for the standard metric on \mathbb{R} .

Solution 34. 1. The map $\psi: E \rightarrow \mathbb{R}_+^*$ defined by $\psi(x) = 1/x$ is a bijection. The metric d is the pullback of the standard metric on \mathbb{R} via ψ :

$$d(x, y) = |\psi(x) - \psi(y)|.$$

Since the standard metric is a metric and ψ is a bijection, d is a metric on E .

2. The open ball is defined as $B_d(a, r) = \{x \in E \mid d(x, a) < r\}$.

- For $B_d(1, 1)$: Solve $\left|\frac{1}{x} - 1\right| < 1$. This gives $0 < \frac{1}{x} < 2$, so $x > \frac{1}{2}$. Thus, $B_d(1, 1) = \left(\frac{1}{2}, \infty\right)$.
- For $B_d\left(\frac{1}{2}, 3\right)$: Solve $\left|\frac{1}{x} - 2\right| < 3$. This gives $-1 < \frac{1}{x} < 5$. Since $x > 0$, we have $0 < \frac{1}{x} < 5$, so $x > \frac{1}{5}$. Thus, $B_d\left(\frac{1}{2}, 3\right) = \left(\frac{1}{5}, \infty\right)$.

3. For a fixed center $x_0 \in E$ and radius $r > 0$, the condition $x \in B_d(x_0, r)$ is

$$\left| \frac{1}{x} - \frac{1}{x_0} \right| < r \iff \frac{1}{x_0} - r < \frac{1}{x} < \frac{1}{x_0} + r.$$

We distinguish two cases.

Case 1: $r \geq \frac{1}{x_0}$. The left inequality is automatically satisfied since $\frac{1}{x} > 0$. The right inequality gives $\frac{1}{x} < \frac{1}{x_0} + r$, so

$$x > \frac{x_0}{1 + rx_0}.$$

Hence, $B_d(x_0, r) = \left(\frac{x_0}{1+rx_0}, \infty\right)$.

Case 2: $r < \frac{1}{x_0}$. Both bounds are positive, and taking reciprocals (which reverses inequalities) yields

$$\frac{x_0}{1 + rx_0} < x < \frac{x_0}{1 - rx_0}.$$

Hence, $B_d(x_0, r) = \left(\frac{x_0}{1+rx_0}, \frac{x_0}{1-rx_0}\right)$.

In summary,

$$B_d(x_0, r) = \begin{cases} \left(\frac{x_0}{1 + rx_0}, \infty\right) & \text{if } r \geq \frac{1}{x_0}, \\ \left(\frac{x_0}{1 + rx_0}, \frac{x_0}{1 - rx_0}\right) & \text{if } r < \frac{1}{x_0}. \end{cases}$$

4. The set $(0, 1]$ is **unbounded** in (E, d) . Consider the sequence $x_n = 1/n \in (0, 1]$. Then $d(x_n, 1) = |n - 1| \rightarrow \infty$ as $n \rightarrow \infty$.

Closedness : Let (x_n) be a sequence in $A = (0, 1]$ that converges to a limit $\ell \in \mathbb{R}_+^*$ with respect to the metric d . By definition of d , this means

$$d(x_n, \ell) = \left| \frac{1}{x_n} - \frac{1}{\ell} \right| \xrightarrow{n \rightarrow \infty} 0.$$

Thus, the sequence $(1/x_n)$ converges to $1/\ell$ in the standard topology of \mathbb{R} .

Since $x_n \in (0, 1]$ for all n , it follows that $1/x_n \in [1, \infty)$ for all n . The set $[1, \infty)$ is closed in \mathbb{R} with the standard topology, so its limit $1/\ell$ must also belong to $[1, \infty)$. Therefore, $1/\ell \geq 1$, which implies $\ell \leq 1$. As $\ell \in \mathbb{R}_+^*$, we have $\ell > 0$, so $\ell \in (0, 1] = A$.

This shows that A contains all its limit points in the metric space (\mathbb{R}_+^*, d) , and hence A is closed.

5. The set $A = (1, \infty)$ is bounded in (E, d) but unbounded in the standard metric. For any $x, y \in A$, we have

$$d(x, y) = \left| \frac{1}{x} - \frac{1}{y} \right| < 1,$$

so $\text{diam}_d(A) \leq 1$. However, A is clearly unbounded in the standard metric on \mathbb{R} .

Exercise 35. Let E be a set, and define $d: E \times E \rightarrow \mathbb{R}$ by

$$d(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y. \end{cases}$$

1. Show that d is a metric on E .
2. Determine the open ball $B(x, r)$ for $x \in E$ and $r > 0$.
3. Determine the open and closed sets in the metric space (E, d) .

Solution 35. 1. We verify the axioms of a metric.

- *Non-negativity*: By definition, $d(x, y) \in \{0, 1\}$, so $d(x, y) \geq 0$ for all $x, y \in E$.
- *Identity of indiscernibles*: $d(x, y) = 0$ if and only if $x = y$, which is true by the definition of d .
- *Symmetry*: If $x = y$, then $d(x, y) = d(y, x) = 0$. If $x \neq y$, then $d(x, y) = d(y, x) = 1$. Thus, $d(x, y) = d(y, x)$ for all $x, y \in E$.
- *Triangle inequality*: Let $x, y, z \in E$. If $x = z$, then $d(x, z) = 0 \leq d(x, y) + d(y, z)$. If $x \neq z$, then $d(x, z) = 1$. In this case, it is impossible for both $x = y$ and $y = z$ to hold (as this would imply $x = z$), so at least one of $d(x, y)$ or $d(y, z)$ is equal to 1. Hence, $d(x, y) + d(y, z) \geq 1 = d(x, z)$.

Therefore, d is a metric on E , called the *discrete metric*.

2. The open ball of center $x \in E$ and radius $r > 0$ is defined as

$$B(x, r) = \{y \in E \mid d(x, y) < r\}.$$

- If $0 < r \leq 1$, then $d(x, y) < r$ implies $d(x, y) = 0$, so $y = x$. Thus, $B(x, r) = \{x\}$.
 - If $r > 1$, then for all $y \in E$, $d(x, y) \leq 1 < r$, so $B(x, r) = E$.
3. Let $A \subseteq E$ be an arbitrary subset. For any $x \in A$, the open ball $B(x, 1/2) = \{x\}$ is contained in A . Therefore, A is a neighborhood of each of its points, which by Theorem 1.8 implies that A is open.

Since every subset of E is open, the complement of any subset is also open. By definition, a set is closed if its complement is open, so every subset of E is also closed. The topology induced by the discrete metric is the *discrete topology*.

Exercise 36. Let $\mathbb{R}^{\mathbb{N}}$ denote the set of all real sequences. For $u = (u_n)_{n \in \mathbb{N}}$ and $v = (v_n)_{n \in \mathbb{N}}$ in $\mathbb{R}^{\mathbb{N}}$, define

$$d(u, v) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|u_n - v_n|}{|u_n - v_n| + 1}.$$

1. Show that d is a metric on $\mathbb{R}^{\mathbb{N}}$.
2. Show that the metric space $(\mathbb{R}^{\mathbb{N}}, d)$ is bounded.

Solution 36. 1. We verify the axioms of a metric.

Well-definedness: For each n , the term $\frac{|u_n - v_n|}{|u_n - v_n| + 1}$ is in $[0, 1)$. Since $\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$, the series for $d(u, v)$ converges by the comparison test, so d is well-defined and $d(u, v) \in [0, 1)$.

Non-negativity and separation: Each term in the sum is non-negative, so $d(u, v) \geq 0$. Moreover, $d(u, v) = 0$ if and only if every term in the sum is zero, which happens if and only if $|u_n - v_n| = 0$ for all n , i.e., $u = v$.

Symmetry: The expression $|u_n - v_n|$ is symmetric in u and v , so $d(u, v) = d(v, u)$.

Triangle inequality: Consider the function $\varphi: [0, \infty) \rightarrow [0, 1)$ defined by $\varphi(t) = \frac{t}{t+1}$. This function is increasing and subadditive (as shown in previous exercises). For each n , the standard metric on \mathbb{R} gives $|u_n - w_n| \leq |u_n - v_n| + |v_n - w_n|$. Applying the increasing function φ and its subadditivity, we get

$$\varphi(|u_n - w_n|) \leq \varphi(|u_n - v_n| + |v_n - w_n|) \leq \varphi(|u_n - v_n|) + \varphi(|v_n - w_n|).$$

Multiplying by $1/2^n$ and summing over n yields $d(u, w) \leq d(u, v) + d(v, w)$.

Therefore, d is a metric on $\mathbb{R}^{\mathbb{N}}$.

2. For any $u, v \in \mathbb{R}^{\mathbb{N}}$, we have

$$d(u, v) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|u_n - v_n|}{|u_n - v_n| + 1} < \sum_{n=1}^{\infty} \frac{1}{2^n} = 1.$$

Thus, the diameter of the space is at most 1, and the metric space $(\mathbb{R}^{\mathbb{N}}, d)$ is bounded.

Exercise 37. Let (E, d) be a metric space. For two subsets $A, B \subseteq E$, define

$$d(A, B) = \inf\{d(a, b) \mid a \in A, b \in B\}, \quad \delta(A) = \sup\{d(a, a') \mid a, a' \in A\}.$$

1. Show that if $A \subseteq B$, then $\delta(A) \leq \delta(B)$.
2. Show that $\delta(A) = \delta(\overline{A})$.
3. Show that $\delta(A \cup B) \leq \delta(A) + \delta(B) + d(A, B)$.

Solution 37. 1. If $A \subseteq B$, then every pair of points in A is also a pair of points in B . Therefore, the set $\{d(a, a') \mid a, a' \in A\}$ is a subset of $\{d(b, b') \mid b, b' \in B\}$. The supremum of a subset cannot exceed the supremum of the whole set, so $\delta(A) \leq \delta(B)$.

2. Since $A \subseteq \overline{A}$, part 1 gives $\delta(A) \leq \delta(\overline{A})$. For the reverse inequality, let $x, y \in \overline{A}$. By the definition of closure, there exist sequences $(x_n), (y_n) \subseteq A$ such that $x_n \rightarrow x$ and $y_n \rightarrow y$. The metric d is continuous, so $d(x_n, y_n) \rightarrow d(x, y)$. Since $d(x_n, y_n) \leq \delta(A)$ for all n , it follows that $d(x, y) \leq \delta(A)$. As this holds for all $x, y \in \overline{A}$, we conclude $\delta(\overline{A}) \leq \delta(A)$. Hence, $\delta(A) = \delta(\overline{A})$.

3. Let $x, y \in A \cup B$. We consider three cases.

- If $x, y \in A$, then $d(x, y) \leq \delta(A) \leq \delta(A) + \delta(B) + d(A, B)$.
- If $x, y \in B$, then $d(x, y) \leq \delta(B) \leq \delta(A) + \delta(B) + d(A, B)$.
- If $x \in A$ and $y \in B$ (or vice versa), then for any $\varepsilon > 0$, there exist $a \in A$ and $b \in B$ such that $d(a, b) < d(A, B) + \varepsilon$. By the triangle inequality,

$$d(x, y) \leq d(x, a) + d(a, b) + d(b, y) \leq \delta(A) + (d(A, B) + \varepsilon) + \delta(B).$$

Since $\varepsilon > 0$ is arbitrary, we get $d(x, y) \leq \delta(A) + \delta(B) + d(A, B)$.

In all cases, the distance between any two points in $A \cup B$ is bounded above by $\delta(A) + \delta(B) + d(A, B)$. Taking the supremum over all such pairs yields the desired inequality.

Exercise 38. Let (E, d) be a metric space and let $A \subseteq E$.

1. For $r > 0$, define the r -open neighborhood of A by

$$V_r(A) = \{x \in E \mid d(x, A) < r\}.$$

- (a) Show that $V_r(A)$ is open in (E, d) .
- (b) Show that if A is closed, then $A = \bigcap_{n=1}^{\infty} V_{1/n}(A)$.

2. Let $\tilde{A} = \bigcap_{n=1}^{\infty} \{x \in E \mid d(x, A) < \frac{1}{n}\}$.

- (a) Show that \tilde{A} is closed.

- (b) Show that \tilde{A} is the smallest closed set containing A (i.e., $\tilde{A} = \bar{A}$).
3. Show that if (E, d) is complete, then every nonempty closed subset is a countable intersection of open sets (i.e., every closed set is a G_δ set).
4. Let $B \neq E$ be a closed subset of (E, d) . Show that B is a countable intersection of open sets (even if E is not complete).
5. Let $\hat{B} = \bigcup_{n=1}^{\infty} \left\{ x \in E \mid d(x, B^c) \geq \frac{1}{n} \right\}$.
- (a) Show that \hat{B} is open in (E, d) .
- (b) Show that \hat{B} is the largest open set contained in B (i.e., $\hat{B} = \overset{\circ}{B}$).

- Solution 38.**
1. (a) Let $x \in V_r(A)$, so $d(x, A) < r$. Choose $\varepsilon > 0$ such that $d(x, A) + \varepsilon < r$. For any $y \in B(x, \varepsilon)$, the triangle inequality gives $d(y, A) \leq d(y, x) + d(x, A) < \varepsilon + d(x, A) < r$. Thus, $B(x, \varepsilon) \subseteq V_r(A)$, so $V_r(A)$ is open.
- (b) Suppose A is closed. Clearly, $A \subseteq V_{1/n}(A)$ for all n , so $A \subseteq \bigcap_{n=1}^{\infty} V_{1/n}(A)$. For the reverse inclusion, let $x \in \bigcap_{n=1}^{\infty} V_{1/n}(A)$. Then $d(x, A) < 1/n$ for all n , so $d(x, A) = 0$. This means there is a sequence in A converging to x . Since A is closed, $x \in A$. Hence, the equality holds.
2. Note that $\tilde{A} = \bigcap_{n=1}^{\infty} V_{1/n}(A)$.
- (a) Each $V_{1/n}(A)$ is open by part 1(a), but \tilde{A} is an intersection of open sets, which is not necessarily open. However, we will show it is closed by proving it equals \bar{A} . Alternatively, one can show its complement is open: if $x \notin \tilde{A}$, then $d(x, A) \geq 1/n$ for some n , so $B(x, 1/(2n))$ is disjoint from \tilde{A} .
- (b) We claim $\tilde{A} = \bar{A}$. Let $x \in \bar{A}$. Then $d(x, A) = 0$, so $x \in V_{1/n}(A)$ for all n , hence $x \in \tilde{A}$. Conversely, if $x \in \tilde{A}$, then $d(x, A) < 1/n$ for all n , so $d(x, A) = 0$, which implies $x \in \bar{A}$. Since \bar{A} is the smallest closed set containing A , the result follows.
3. Let $F \subseteq E$ be closed and nonempty. By part 2, $F = \bigcap_{n=1}^{\infty} V_{1/n}(F)$. Each $V_{1/n}(F)$ is open by part 1(a). This representation holds in any metric space; completeness is not required. (Thus, the hypothesis of completeness is superfluous for this conclusion).
4. The proof is the same as in part 3. For any closed set B in a metric space, $B = \bigcap_{n=1}^{\infty} V_{1/n}(B)$, a countable intersection of open sets. The condition $B \neq E$ ensures the sets $V_{1/n}(B)$ are proper, but the equality holds regardless.
5. (a) Let $x \in \hat{B}$, so $d(x, B^c) \geq 1/n$ for some n . Then the open ball $B(x, 1/(2n))$ is contained in \hat{B} , because for any y in this ball, $d(y, B^c) \geq d(x, B^c) - d(x, y) > 1/n - 1/(2n) = 1/(2n) > 0$, so $y \in \hat{B}$. Thus, \hat{B} is open.
- (b) First, $\hat{B} \subseteq B$ because if $x \in \hat{B}$, then $d(x, B^c) > 0$, so $x \notin B^c$, hence $x \in B$. Now let U be any open set with $U \subseteq B$. For any $x \in U$, there exists $\varepsilon > 0$ such that $B(x, \varepsilon) \subseteq U \subseteq B$, which implies $d(x, B^c) \geq \varepsilon$. Choosing n such that $1/n \leq \varepsilon$, we get $x \in \{y \mid d(y, B^c) \geq 1/n\} \subseteq \hat{B}$. Thus, $U \subseteq \hat{B}$, so \hat{B} is the largest open set contained in B , i.e., $\hat{B} = \overset{\circ}{B}$.

Exercise 39. Part 1: Consider the metric space (\mathbb{R}^2, d_2) , where for all $x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathbb{R}^2$, the Euclidean distance is

$$d_2(x, y) = \left(\sum_{i=1}^2 (x_i - y_i)^2 \right)^{1/2}.$$

1. Show that the half-plane $F = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 \leq 0\}$ is closed in \mathbb{R}^2 .
2. Show that any straight line in \mathbb{R}^2 is closed.

Part 2: Define a new metric δ on \mathbb{R}^2 by

$$\delta(x, y) = \max_{1 \leq i \leq 2} |x_i - y_i|.$$

1. Show that δ defines a metric on \mathbb{R}^2 .
2. Describe the open balls of center $C = (1, 1)$ and radius $r > 0$.

Part 3: Define the function $\rho: \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}_+$ by

$$\rho(x, y) = \sum_{i=1}^2 |x_i - y_i|.$$

1. Show that ρ defines a metric on \mathbb{R}^2 .
2. Draw the open ball of center $O = (0, 0)$ and radius 2.

Solution 39. Part 1: The Euclidean metric d_2 .

1. The complement of F is $F^c = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 > 0\}$. Let $p = (p_1, p_2) \in F^c$, so $p_2 > 0$. Consider the open ball $B_{d_2}(p, r)$ with radius $r = p_2/2 > 0$. For any $x = (x_1, x_2) \in B_{d_2}(p, r)$, we have

$$|x_2 - p_2| \leq d_2(x, p) < r = \frac{p_2}{2},$$

which implies $x_2 > p_2 - p_2/2 = p_2/2 > 0$. Thus, $x \in F^c$, so $B_{d_2}(p, r) \subseteq F^c$. Hence, F^c is open, and F is closed.

2. Let $L = \{(x_1, x_2) \in \mathbb{R}^2 \mid ax_1 + bx_2 + c = 0\}$ be a straight line, where $(a, b) \neq (0, 0)$. The distance from a point $p = (p_1, p_2)$ to the line L is given by

$$\text{dist}(p, L) = \frac{|ap_1 + bp_2 + c|}{\sqrt{a^2 + b^2}}.$$

If $p \notin L$, then $\text{dist}(p, L) = d_0 > 0$. The open ball $B_{d_2}(p, d_0/2)$ is disjoint from L , so the complement of L is open. Therefore, L is closed.

Part 2: The Chebyshev metric δ .

1. The function $\delta(x, y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$ is the ℓ^∞ -metric on \mathbb{R}^2 . It is a standard result that this is a metric: non-negativity, symmetry, and the identity of indiscernibles are immediate. The triangle inequality follows from the fact that the maximum of a sum is less than or equal to the sum of the maxima.
2. The open ball of center $C = (1, 1)$ and radius $r > 0$ is

$$B_\delta(C, r) = \{(x, y) \in \mathbb{R}^2 \mid \max\{|x - 1|, |y - 1|\} < r\} =]1 - r, 1 + r[\times]1 - r, 1 + r[,$$

which is an open square (with sides parallel to the axes) centered at $(1, 1)$.

Part 3: The Manhattan metric ρ .

1. The function $\rho(x, y) = |x_1 - y_1| + |x_2 - y_2|$ is the ℓ^1 -metric on \mathbb{R}^2 . It satisfies all the metric axioms: non-negativity, symmetry, and the identity of indiscernibles are clear. The triangle inequality holds because the absolute value satisfies the triangle inequality, and the sum of inequalities is an inequality.
2. The open ball of center $O = (0, 0)$ and radius 2 is

$$B_\rho(O, 2) = \{(x, y) \in \mathbb{R}^2 \mid |x| + |y| < 2\}.$$

This is a diamond-shaped region (a square rotated by 45°) with vertices at $(2, 0)$, $(0, 2)$, $(-2, 0)$, and $(0, -2)$.

1.12.4 Topology Associated with a Metric

Proposition 1.96. Let (E, d) be a metric space, and let $O \subseteq E$ be a non-empty subset. The following properties are equivalent:

1. O is a union of open balls.
2. For every $x \in O$, there exists $r > 0$ such that $B(x, r) \subseteq O$.

Proof. ($2 \Rightarrow 1$) For each $x \in O$, choose $r_x > 0$ such that $B(x, r_x) \subseteq O$. Then

$$O = \bigcup_{x \in O} B(x, r_x),$$

since every x is in its own ball, and every ball is contained in O .

($1 \Rightarrow 2$) Suppose $O = \bigcup_{i \in I} B(x_i, r_i)$. For any $x \in O$, there exists an index i such that $x \in B(x_i, r_i)$. Let $r = r_i - d(x, x_i) > 0$. By the triangle inequality, $B(x, r) \subseteq B(x_i, r_i) \subseteq O$, which establishes the claim. ■

Theorem 1.97. The collection of all subsets of E satisfying the equivalent properties of the previous proposition forms a topology on E . This topology is called the *topology induced by the metric d* .

Proof. Let \mathcal{T}_d be the collection of all subsets of E that are unions of open balls.

1. *The empty set and the whole space are in \mathcal{T}_d .* The empty set is the empty union of balls, so $\emptyset \in \mathcal{T}_d$. The whole space is $E = \bigcup_{x \in E} B(x, 1)$, so $E \in \mathcal{T}_d$.
2. *\mathcal{T}_d is closed under finite intersections.* Let $O_1, \dots, O_k \in \mathcal{T}_d$ and let $x \in \bigcap_{i=1}^k O_i$. For each i , there exists $r_i > 0$ such that $B(x, r_i) \subseteq O_i$. Let $r = \min\{r_1, \dots, r_k\} > 0$. Then $B(x, r) \subseteq \bigcap_{i=1}^k O_i$. By the proposition, the intersection is a union of open balls, so it belongs to \mathcal{T}_d .
3. *\mathcal{T}_d is closed under arbitrary unions.* Let $\{O_i\}_{i \in I} \subseteq \mathcal{T}_d$. Each O_i is a union of open balls, so their total union $\bigcup_{i \in I} O_i$ is also a union of open balls, hence in \mathcal{T}_d .

Therefore, \mathcal{T}_d is a topology on E . ■

Definition 1.98 (Metrisable Spaces). A topological space (E, \mathcal{T}) is said to be *metrisable* if there exists a metric d on E such that the topology \mathcal{T}_d induced by d coincides with \mathcal{T} .

Example 1.99. 1. **The discrete metric induces the discrete topology.** Let d be the discrete metric on E . For any $x \in E$, the open ball $B(x, 1/2) = \{x\}$. Thus, every singleton is open, and since any subset is a union of singletons, every subset of E is open. Hence, the induced topology is the discrete topology.

2. **The standard metric induces the usual topology on \mathbb{R} .** The open balls for the standard metric $d(x, y) = |x - y|$ are the open intervals $]x - r, x + r[$. Since the usual topology on \mathbb{R} is generated by these intervals, the two topologies coincide.

There are other metrics that induce the same topology. For example, the metric $d(x, y) = |\arctan x - \arctan y|$ is topologically equivalent to the standard metric, as the function $\arctan: \mathbb{R} \rightarrow]-\pi/2, \pi/2[$ is a homeomorphism.

1.12.5 Lipschitz and Contraction Mappings

Definition 1.100. Let (E, d) and (E', d') be two metric spaces, and let $f: E \rightarrow E'$ be a function. The map f is said to be k -Lipschitz (or *Lipschitz continuous with constant k*) for some real number $k \geq 0$ if

$$\forall x, y \in E, \quad d'(f(x), f(y)) \leq k d(x, y).$$

If such a constant k exists, the smallest such constant is called the *Lipschitz constant* of f (or the *Lipschitz ratio*).

- The function f is *bilipschitz* if it is bijective and both f and its inverse f^{-1} are Lipschitz.
- The function f is a *contraction mapping* if it is Lipschitz with a constant $k < 1$.

Example 1.101. Consider the real line \mathbb{R} with its standard metric $d(x, y) = |x - y|$, and define $f: \mathbb{R} \rightarrow \mathbb{R}$ by

$$f(x) = \frac{2}{7}|x|.$$

For all $x, y \in \mathbb{R}$, we have

$$|f(x) - f(y)| = \frac{2}{7}||x| - |y|| \leq \frac{2}{7}|x - y|,$$

where the inequality follows from the reverse triangle inequality. Thus, f is $\frac{2}{7}$ -Lipschitz, and its Lipschitz constant is $k = \frac{2}{7}$.

Definition 1.102 (Isometries). Let (E, d) and (E', d') be metric spaces. A function $f: E \rightarrow E'$ is an *isometry* if

$$\forall x, y \in E, \quad d'(f(x), f(y)) = d(x, y).$$

In other words, an isometry is a distance-preserving map. Every isometry is injective, 1-Lipschitz, and a homeomorphism onto its image. If it is also surjective, it is a global homeomorphism.

Example 1.103 (Isometry between \mathbb{R}^2 and \mathbb{C}). Equip \mathbb{C} with the standard metric $d_{\mathbb{C}}(z_1, z_2) = |z_1 - z_2|$, and \mathbb{R}^2 with the Euclidean metric $d_2((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$. The map

$$f: \mathbb{R}^2 \rightarrow \mathbb{C}, \quad f(x, y) = x + iy$$

is an isometry, since

$$d_{\mathbb{C}}(f(x_1, y_1), f(x_2, y_2)) = |(x_1 - x_2) + i(y_1 - y_2)| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = d_2((x_1, y_1), (x_2, y_2)).$$

Proposition 1.104. Let (E, d) be a metric space and let $A \subseteq E$ be a non-empty subset. The function

$$\varphi_A: E \rightarrow \mathbb{R}, \quad \varphi_A(x) = d(x, A) = \inf_{a \in A} d(x, a)$$

is 1-Lipschitz. That is, for all $x, y \in E$,

$$|d(x, A) - d(y, A)| \leq d(x, y).$$

Proof. Fix $x, y \in E$. For any $a \in A$, the triangle inequality gives

$$d(x, a) \leq d(x, y) + d(y, a).$$

Taking the infimum over $a \in A$ on both sides yields

$$d(x, A) \leq d(x, y) + d(y, A).$$

Rearranging, we obtain $d(x, A) - d(y, A) \leq d(x, y)$. By symmetry, interchanging x and y gives $d(y, A) - d(x, A) \leq d(x, y)$. Combining these two inequalities,

$$|d(x, A) - d(y, A)| \leq d(x, y),$$

as required. ■

1.12.6 Topologically Equivalent and Equivalent Metrics

Definition 1.105. Let E be a set, and let d and d' be two metrics on E .

- The metrics d and d' are *topologically equivalent* if they induce the same topology on E , i.e., $\mathcal{T}_d = \mathcal{T}_{d'}$. This means that a subset of E is open with respect to d if and only if it is open with respect to d' .
- The metrics d and d' are (*metrically*) *equivalent* if there exist positive constants $\alpha, \beta > 0$ such that for all $x, y \in E$,

$$\alpha d'(x, y) \leq d(x, y) \leq \beta d'(x, y).$$

Example 1.106 (Topological equivalence of the Euclidean and Manhattan metrics). On \mathbb{R}^n , consider the following two metrics:

- The *Euclidean metric*:

$$d(x, y) = \sqrt{\sum_{i=1}^n |x_i - y_i|^2},$$

- The *Manhattan metric* (or ℓ^1 -metric):

$$d'(x, y) = \sum_{i=1}^n |x_i - y_i|,$$

where $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$.

The open ball of radius $\varepsilon > 0$ centered at a point x is:

- For the Euclidean metric: $B_d(x, \varepsilon) = \{y \in \mathbb{R}^n \mid d(x, y) < \varepsilon\}$, which is a spherical (round) ball.
- For the Manhattan metric: $B_{d'}(x, \varepsilon) = \{y \in \mathbb{R}^n \mid d'(x, y) < \varepsilon\}$, which is a diamond-shaped (polyhedral) ball.

These two metrics are *metrically equivalent*, and hence topologically equivalent. Indeed, for all $x, y \in \mathbb{R}^n$, the following inequalities hold:

$$\frac{1}{\sqrt{n}} d'(x, y) \leq d(x, y) \leq d'(x, y).$$

The right-hand inequality follows from $|x_i - y_i| \leq d(x, y)$ and summing over i . The left-hand inequality is a consequence of the Cauchy–Schwarz inequality:

$$d'(x, y) = \sum_{i=1}^n |x_i - y_i| \cdot 1 \leq \left(\sum_{i=1}^n |x_i - y_i|^2 \right)^{1/2} \left(\sum_{i=1}^n 1^2 \right)^{1/2} = \sqrt{n} d(x, y).$$

This mutual bounding by constant multiples implies that every open ball in one metric contains an open ball of the other metric, and vice versa. Consequently, the two metrics induce the exact same topology on \mathbb{R}^n .

Remark 1.107. 1. If two metrics are metrically equivalent, then they are topologically equivalent.

2. The converse is false: there exist metrics that are topologically equivalent but not metrically equivalent.

Example 1.108 (Topologically and metrically equivalent metrics on \mathbb{R}^n). On \mathbb{R}^n , the following metrics are metrically (hence topologically) equivalent:

- *Manhattan metric:* $d_1(x, y) = \sum_{i=1}^n |x_i - y_i|$,
- *Euclidean metric:* $d_2(x, y) = (\sum_{i=1}^n |x_i - y_i|^2)^{1/2}$,
- *Chebyshev metric:* $d_\infty(x, y) = \max_{1 \leq i \leq n} |x_i - y_i|$.

They satisfy the inequalities

$$d_\infty(x, y) \leq d_2(x, y) \leq d_1(x, y) \leq \sqrt{n} d_2(x, y) \leq n d_\infty(x, y),$$

which establish their metric equivalence.

Example 1.109 (Topologically but not metrically equivalent metrics on \mathbb{R}). Let $d(x, y) = |x - y|$ be the standard metric on \mathbb{R} , and let

$$d'(x, y) = \frac{|x - y|}{1 + |x - y|}.$$

The metrics d and d' are topologically equivalent because $d(x_n, x) \rightarrow 0$ if and only if $d'(x_n, x) \rightarrow 0$. However, they are not metrically equivalent. For metric equivalence, there would need to exist $\beta > 0$ such that $d(x, y) \leq \beta d'(x, y)$ for all x, y . But as $|x - y| \rightarrow \infty$, $d'(x, y) \rightarrow 1$ while $d(x, y) \rightarrow \infty$, making this impossible.

Proposition 1.110. Let E be a non-empty set, and let d and d' be two metrics on E .

1. The metrics d and d' are *metrically equivalent* if and only if the identity map $\text{Id}_E: (E, d) \rightarrow (E, d')$ is bilipschitz.
2. The metrics d and d' are *topologically equivalent* if and only if the identity map $\text{Id}_E: (E, d) \rightarrow (E, d')$ is a homeomorphism.

Proof. 1. The identity map $\text{Id}_E: (E, d) \rightarrow (E, d')$ is Lipschitz if and only if there exists a constant $k > 0$ such that

$$d'(x, y) \leq k d(x, y) \quad \text{for all } x, y \in E.$$

Its inverse, which is also the identity map $\text{Id}_E: (E, d') \rightarrow (E, d)$, is Lipschitz if and only if there exists a constant $k' > 0$ such that

$$d(x, y) \leq k' d'(x, y) \quad \text{for all } x, y \in E.$$

Combining these two inequalities, we obtain

$$\frac{1}{k'} d(x, y) \leq d'(x, y) \leq k d(x, y) \quad \text{for all } x, y \in E,$$

which is precisely the definition of metric equivalence. Therefore, Id_E is bilipschitz if and only if d and d' are metrically equivalent.

2. The identity map $\text{Id}_E: (E, d) \rightarrow (E, d')$ is continuous if and only if the preimage of every d' -open set is d -open. Since the map is the identity, this means that every d' -open set is also d -open, i.e., $\mathcal{T}_{d'} \subseteq \mathcal{T}_d$.

Similarly, the inverse map $\text{Id}_E: (E, d') \rightarrow (E, d)$ is continuous if and only if $\mathcal{T}_d \subseteq \mathcal{T}_{d'}$.

The identity map is a homeomorphism if and only if it is continuous and its inverse is continuous, which is equivalent to $\mathcal{T}_d = \mathcal{T}_{d'}$. This is exactly the definition of topological equivalence. ■

Exercise 40. Let (E, d) be a metric space, and define a new function $d': E \times E \rightarrow \mathbb{R}_+$ by

$$d'(x, y) = \frac{d(x, y)}{1 + d(x, y)}, \quad \text{for all } x, y \in E.$$

1. Show that d' is a valid metric on E and that it is bounded.
2. Show that d and d' are topologically equivalent.

Solution 40. 1. We verify the axioms of a metric for d' .

- *Non-negativity and separation:* Since $d(x, y) \geq 0$, we have $d'(x, y) \geq 0$. Moreover, $d'(x, y) = 0$ if and only if $d(x, y) = 0$, which is equivalent to $x = y$.
- *Symmetry:* $d'(x, y) = \frac{d(x, y)}{1 + d(x, y)} = \frac{d(y, x)}{1 + d(y, x)} = d'(y, x)$.
- *Triangle inequality:* Consider the function $\varphi: [0, \infty) \rightarrow [0, 1)$ defined by $\varphi(t) = \frac{t}{1+t}$. This function is increasing and *subadditive*, meaning $\varphi(a + b) \leq \varphi(a) + \varphi(b)$ for all $a, b \geq 0$. Since d is a metric, $d(x, z) \leq d(x, y) + d(y, z)$. As φ is increasing and subadditive, we have

$$d'(x, z) = \varphi(d(x, z)) \leq \varphi(d(x, y) + d(y, z)) \leq \varphi(d(x, y)) + \varphi(d(y, z)) = d'(x, y) + d'(y, z).$$

Therefore, d' is a metric. It is bounded because $d'(x, y) < 1$ for all $x, y \in E$.

2. To prove topological equivalence, we show that the identity map $\text{Id}: (E, d) \rightarrow (E, d')$ is a homeomorphism. It suffices to show that a sequence (x_n) converges to x in (E, d) if and only if it converges to x in (E, d') .

Since the function $\varphi(t) = t/(1+t)$ is continuous at 0 with $\varphi(0) = 0$, and its inverse $\varphi^{-1}(s) = s/(1-s)$ is also continuous at 0, we have:

$$d(x_n, x) \xrightarrow[n \rightarrow \infty]{} 0 \quad \iff \quad d'(x_n, x) = \varphi(d(x_n, x)) \xrightarrow[n \rightarrow \infty]{} 0.$$

This shows that the two metrics have the same convergent sequences, and therefore induce the same topology on E .

1.12.7 Sequential Characterizations

Let (E, d) be a metric space. Recall that for any point $x \in E$, the family of open balls $\{B(x, \varepsilon) \mid \varepsilon > 0\}$ forms a neighborhood basis at x . Using this, the definition of the limit of a sequence in a general topological space specializes to the following precise analytical formulation in metric spaces.

Proposition 1.111. Let (E, d) be a metric space and let $(x_n)_{n \in \mathbb{N}}$ be a sequence of elements in E . Then the sequence (x_n) converges to a limit $\ell \in E$ if and only if

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ such that } \forall n \in \mathbb{N}, n \geq N \implies d(x_n, \ell) < \varepsilon.$$

Proof. (\implies) Assume (x_n) converges to ℓ in the topological sense. Let $\varepsilon > 0$ be given. The open ball $B(\ell, \varepsilon)$ is a neighborhood of ℓ . By the definition of convergence, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $x_n \in B(\ell, \varepsilon)$, which is equivalent to $d(x_n, \ell) < \varepsilon$.

(\impliedby) Conversely, suppose the ε - N condition holds. Let V be an arbitrary neighborhood of ℓ . By the definition of the metric topology, there exists $r > 0$ such that $B(\ell, r) \subseteq V$. Choose $\varepsilon = r/2 > 0$. By hypothesis, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $d(x_n, \ell) < \varepsilon$. This implies $x_n \in B(\ell, \varepsilon) \subseteq B(\ell, r) \subseteq V$. Since this holds for every neighborhood V of ℓ , the sequence (x_n) converges to ℓ .

Finally, since every metric space is Hausdorff, the limit of a convergent sequence is unique. ■

1.12.8 Adherent Points and Closed Sets

Definition 1.112. Let (E, d) be a metric space, let $A \subseteq E$, and let $a \in E$. The point a is said to be *adherent* to A if

$$\forall \varepsilon > 0, B(a, \varepsilon) \cap A \neq \emptyset.$$

Equivalently, a is adherent to A if and only if

$$\forall \varepsilon > 0, \exists x \in A \text{ such that } d(x, a) < \varepsilon.$$

The set of all points adherent to A is precisely the *closure* of A , denoted \overline{A} .

Example 1.113. Consider the set $A =]0, 1[\subset \mathbb{R}$, where \mathbb{R} is equipped with its standard metric. The points 0 and 1 are adherent to A .

Indeed, for any $\varepsilon > 0$, the open ball (interval) $B(0, \varepsilon) =]-\varepsilon, \varepsilon[$ intersects A because, for instance, $\varepsilon/2 \in A \cap B(0, \varepsilon)$. Similarly, $B(1, \varepsilon) =]1 - \varepsilon, 1 + \varepsilon[$ intersects A since $1 - \varepsilon/2 \in A \cap B(1, \varepsilon)$. Hence, both 0 and 1 are adherent points of A , and in fact $\overline{A} = [0, 1]$.

Proposition 1.114. (Characterization of the Closure)

Let (E, d) be a metric space, let $A \subseteq E$ be a non-empty subset, and let $x \in E$. The following three statements are equivalent:

1. $x \in \overline{A}$ (the closure of A).
2. $d(x, A) = 0$.
3. There exists a sequence $(a_n)_{n \in \mathbb{N}}$ of elements of A that converges to x .

Proof. We prove the equivalence by showing the implications $(1) \implies (2) \implies (3) \implies (1)$.

(1) \implies (2): Assume $x \in \overline{A}$. By definition of the closure, for every $\varepsilon > 0$, the open ball $B(x, \varepsilon)$ intersects A . In particular, for every $n \in \mathbb{N}^*$, there exists a point $a_n \in A$ such that $d(x, a_n) < 1/n$. This implies that $d(x, A) \leq 1/n$ for all n . Taking the limit as $n \rightarrow \infty$, we obtain $d(x, A) = 0$.

(2) \Rightarrow (3): Assume $d(x, A) = 0$. By the definition of the infimum, for every $n \in \mathbb{N}^*$, there exists a point $a_n \in A$ such that $d(x, a_n) < 1/n$. The sequence $(a_n)_{n \in \mathbb{N}}$ is therefore a sequence in A that converges to x .

(3) \Rightarrow (1): Assume there is a sequence $(a_n) \subseteq A$ such that $a_n \rightarrow x$. Let $\varepsilon > 0$ be arbitrary. By the definition of convergence, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $d(a_n, x) < \varepsilon$. In particular, $a_N \in B(x, \varepsilon) \cap A$, so this intersection is non-empty. Since this holds for every $\varepsilon > 0$, x is adherent to A , which means $x \in \bar{A}$. ■

Remark 1.115. The set of adherent points of a subset $A \subseteq E$ can be partitioned into two distinct types:

1. A point $x \in E$ is an *accumulation point* (or *limit point*) of A if every neighborhood of x contains at least one point of A different from x itself. Equivalently, for all $r > 0$,

$$(B(x, r) \setminus \{x\}) \cap A \neq \emptyset.$$

(Note: Some authors include x itself in the intersection, but the standard definition requires points *other than* x).

2. A point $x \in A$ is an *isolated point* of A if there exists a radius $r > 0$ such that

$$B(x, r) \cap A = \{x\}.$$

That is, x is in A but is not an accumulation point of A .

The closure of A is the disjoint union of its set of accumulation points and its set of isolated points.

Proposition 1.116. (Sequential Characterization of Closed Sets)

Let (E, d) be a metric space, and let $A \subseteq E$. The set A is closed if and only if it is *sequentially closed*, that is, for every sequence $(a_n)_{n \in \mathbb{N}}$ of elements of A , if (a_n) converges to a limit $x \in E$, then $x \in A$.

Proof. (\Rightarrow) Suppose A is closed, so $A = \bar{A}$. Let $(a_n) \subseteq A$ be a sequence such that $a_n \rightarrow x \in E$. By Proposition ??, $x \in \bar{A}$. Since $A = \bar{A}$, it follows that $x \in A$.

(\Leftarrow) Conversely, assume that A is sequentially closed. We will show that $A = \bar{A}$. The inclusion $A \subseteq \bar{A}$ is always true. To prove the reverse inclusion, let $x \in \bar{A}$. By Proposition ??, there exists a sequence $(a_n) \subseteq A$ such that $a_n \rightarrow x$. By the sequential closedness of A , we must have $x \in A$. Thus, $\bar{A} \subseteq A$, and so $A = \bar{A}$, which means A is closed. ■

Exercise 41. (Distance to a Closed Set)

Let F be a closed subset of a metric space (E, d) . Assume that $d(x, F) = 0$. Prove that $x \in F$.

Solution 41. Since $d(x, F) = 0$, by the definition of the infimum, for every $\varepsilon > 0$ there exists a point $y \in F$ such that $d(x, y) < \varepsilon$.

In particular, for each $n \in \mathbb{N}^*$, choose a point $x_n \in F$ such that

$$d(x, x_n) < \frac{1}{n}.$$

The sequence $(x_n)_{n \in \mathbb{N}}$ is then a sequence of elements in F that converges to x , since $d(x, x_n) \rightarrow 0$ as $n \rightarrow \infty$.

Because F is closed, it contains the limit of every convergent sequence of its points. Therefore, $x \in F$.

1.12.9 Uniform Continuity

Definition 1.117. Let (E, d) and (E', δ) be metric spaces, and let $f: E \rightarrow E'$ be a function.

1. The function f is *continuous at a point* $x_0 \in E$ if

$$\forall \varepsilon > 0, \exists \eta > 0, \forall x \in E, d(x, x_0) < \eta \implies \delta(f(x), f(x_0)) < \varepsilon.$$

2. The function f is *uniformly continuous on* E if

$$\forall \varepsilon > 0, \exists \eta > 0, \forall x, x' \in E, d(x, x') < \eta \implies \delta(f(x), f(x')) < \varepsilon.$$

Remark 1.118. Every uniformly continuous function is continuous at every point of its domain. However, the converse is not true in general. A classic example is the function $f: \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = x^2$, which is continuous everywhere but not uniformly continuous on \mathbb{R} . In fact, among real polynomial functions, only the affine functions (i.e., functions of the form $f(x) = ax + b$) are uniformly continuous on the entire real line.

Example 1.119. (Uniform Continuity and Lipschitz Maps)

1. The function $f: \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = x^2$ is continuous on \mathbb{R} but *not uniformly continuous*.

To see this, fix $\varepsilon = 1$. For any $\eta > 0$, choose $x = \frac{1}{\eta} + \frac{\eta}{2}$ and $y = \frac{1}{\eta}$. Then

$$|x - y| = \frac{\eta}{2} < \eta,$$

but

$$|f(x) - f(y)| = \left| \left(\frac{1}{\eta} + \frac{\eta}{2} \right)^2 - \left(\frac{1}{\eta} \right)^2 \right| = 1 + \frac{\eta^2}{4} > 1 = \varepsilon.$$

Hence, the uniform continuity condition fails.

2. Any Lipschitz continuous mapping is uniformly continuous. Specifically, let $f: (E, d) \rightarrow (E', \delta)$ be a k -Lipschitz function for some $k \geq 0$, i.e.,

$$\delta(f(x), f(y)) \leq k d(x, y) \quad \text{for all } x, y \in E.$$

Then for any $\varepsilon > 0$, choosing $\eta = \varepsilon/k$ (with η arbitrary if $k = 0$) ensures that $d(x, y) < \eta$ implies $\delta(f(x), f(y)) < \varepsilon$. Therefore, f is uniformly continuous on E .

Proposition 1.120. (Composition of Uniformly Continuous Maps)

Let (E, d) , (E', d') , and (E'', d'') be metric spaces, and let $f: E \rightarrow E'$ and $g: E' \rightarrow E''$ be mappings. If f is uniformly continuous on E and g is uniformly continuous on E' , then the composition $g \circ f: E \rightarrow E''$ is uniformly continuous on E .

Proof. Let $\varepsilon > 0$ be given.

Since g is uniformly continuous on E' , there exists $\eta > 0$ such that for all $u, v \in E'$,

$$d'(u, v) < \eta \implies d''(g(u), g(v)) < \varepsilon.$$

Since f is uniformly continuous on E , there exists $\delta > 0$ such that for all $x, y \in E$,

$$d(x, y) < \delta \implies d'(f(x), f(y)) < \eta.$$

Now, let $x, y \in E$ with $d(x, y) < \delta$. Then $d'(f(x), f(y)) < \eta$, and by the uniform continuity of g , it follows that

$$d''((g \circ f)(x), (g \circ f)(y)) = d''(g(f(x)), g(f(y))) < \varepsilon.$$

Thus, for every $\varepsilon > 0$, we have found a $\delta > 0$ such that $d(x, y) < \delta$ implies $d''((g \circ f)(x), (g \circ f)(y)) < \varepsilon$. Therefore, $g \circ f$ is uniformly continuous on E . ■

Exercise 42. (Uniformly Continuous Maps Preserve Cauchy Sequences)

Let $f: (E, d) \rightarrow (F, d')$ be a uniformly continuous function between two metric spaces. Show that f maps every Cauchy sequence in E to a Cauchy sequence in F .

Solution 42. Let $(x_n)_{n \in \mathbb{N}}$ be a Cauchy sequence in (E, d) . We must show that the image sequence $(f(x_n))_{n \in \mathbb{N}}$ is Cauchy in (F, d') .

Let $\varepsilon > 0$ be arbitrary. Since f is uniformly continuous, there exists $\delta > 0$ such that for all $x, y \in E$,

$$d(x, y) < \delta \implies d'(f(x), f(y)) < \varepsilon.$$

Because (x_n) is a Cauchy sequence, there exists an integer $N \in \mathbb{N}$ such that for all $n, p \geq N$,

$$d(x_n, x_p) < \delta.$$

Applying the uniform continuity of f , we obtain for all $n, p \geq N$,

$$d'(f(x_n), f(x_p)) < \varepsilon.$$

This shows that $(f(x_n))$ is a Cauchy sequence in (F, d') , as required.

Exercise 43. Let (X, d_X) and (Y, d_Y) be metric spaces, let $f: X \rightarrow Y$ be a mapping, and let $a \in X$. Show that f is continuous at a if and only if for every sequence $(x_n)_{n \in \mathbb{N}}$ of elements of X that converges to a , the sequence $(f(x_n))_{n \in \mathbb{N}}$ converges to $f(a)$ in Y .

Solution 43. We prove both implications.

(\Rightarrow) Assume that f is continuous at a , and let $(x_n)_{n \in \mathbb{N}}$ be a sequence in X with $x_n \rightarrow a$. We must show that $f(x_n) \rightarrow f(a)$.

Let $\varepsilon > 0$. By continuity of f at a , there exists $\eta > 0$ such that for all $x \in X$,

$$d_X(x, a) < \eta \implies d_Y(f(x), f(a)) < \varepsilon.$$

Since $x_n \rightarrow a$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,

$$d_X(x_n, a) < \eta.$$

For these n , the above implication gives $d_Y(f(x_n), f(a)) < \varepsilon$. This proves that $f(x_n) \rightarrow f(a)$.

(\Leftarrow) We prove the contrapositive. Suppose f is *not* continuous at a . Then there exists $\varepsilon_0 > 0$ such that for every $\eta > 0$, there is some $x \in X$ with

$$d_X(x, a) < \eta \quad \text{and} \quad d_Y(f(x), f(a)) \geq \varepsilon_0.$$

In particular, for each $n \in \mathbb{N}^*$, choose $\eta = 1/n$. There exists a point $x_n \in X$ such that

$$d_X(x_n, a) < \frac{1}{n} \quad \text{and} \quad d_Y(f(x_n), f(a)) \geq \varepsilon_0.$$

The sequence $(x_n)_{n \in \mathbb{N}^*}$ thus satisfies $x_n \rightarrow a$, but the sequence $(f(x_n))$ does not converge to $f(a)$ because its terms are always at least ε_0 away from $f(a)$.

This completes the proof of the equivalence.

1.13 Separable Metric Spaces

Definition 1.121. A metric space (E, d) is said to be *separable* if it contains a countable subset $D \subseteq E$ that is dense in E , i.e., $\overline{D} = E$.

Remark 1.122. It is important not to confuse *separable* with *separated* (Hausdorff). Every metric space is Hausdorff (i.e., for any $x \neq y$, there exist disjoint open balls around x and y), but not every metric space is separable. Separability is a property concerning the *size* (cardinality) of a dense subset, not the separation of points.

Example 1.123. .

1. The real line \mathbb{R} with its standard metric is separable, since the set of rational numbers \mathbb{Q} is countable and dense.
2. More generally, \mathbb{R}^n is separable, with the countable dense subset \mathbb{Q}^n .
3. Any discrete metric space is separable if and only if it is countable. Indeed, in a discrete space, the only dense subset is the entire space itself.

Proposition 1.124. Every metric space is Hausdorff.

Proof. Let (E, d) be a metric space, and let $x, y \in E$ with $x \neq y$. Define $r = d(x, y) > 0$. Consider the open balls

$$V = B\left(x, \frac{r}{3}\right) \quad \text{and} \quad W = B\left(y, \frac{r}{3}\right).$$

Suppose, for contradiction, that there exists a point $z \in V \cap W$. Then

$$d(x, z) < \frac{r}{3} \quad \text{and} \quad d(z, y) < \frac{r}{3}.$$

By the triangle inequality,

$$r = d(x, y) \leq d(x, z) + d(z, y) < \frac{r}{3} + \frac{r}{3} = \frac{2r}{3},$$

which is a contradiction since $r > 0$. Therefore, $V \cap W = \emptyset$, and the space is Hausdorff. ■

Proposition 1.125. In a Hausdorff space, the limit of a convergent sequence is unique.

Proof. Let $(u_n)_{n \in \mathbb{N}}$ be a sequence in a Hausdorff space E , and suppose it has two distinct limits ℓ and ℓ' with $\ell \neq \ell'$. Since E is Hausdorff, there exist disjoint open neighborhoods $V \in \mathcal{V}(\ell)$ and $W \in \mathcal{V}(\ell')$ such that $V \cap W = \emptyset$.

By the definition of convergence, there exist $n_1, n_2 \in \mathbb{N}$ such that for all $n \geq n_1$, $u_n \in V$, and for all $n \geq n_2$, $u_n \in W$. For any $n \geq \max(n_1, n_2)$, we have $u_n \in V \cap W = \emptyset$, a contradiction. Hence, the limit must be unique. ■