CONDUCTORS IN ELECTROSTATIC EQUILIBRIUM

OBJECTIVES

→ Establish the electrostatic properties of conductors: field, potential, and energy.

→ Study the properties of a system of two conductors under mutual influence.

→ Apply these results to capacitors: properties, capacitance, forces on the plates.

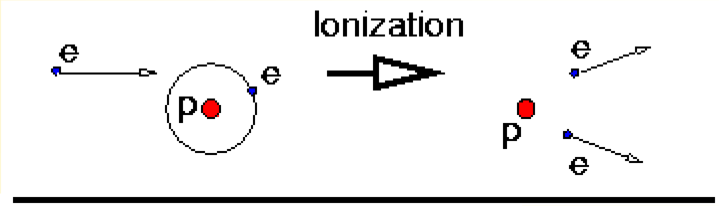
I- CONDUCTOR IN ELECTROSTATIC EQUILIBRIUM

1- Definitions

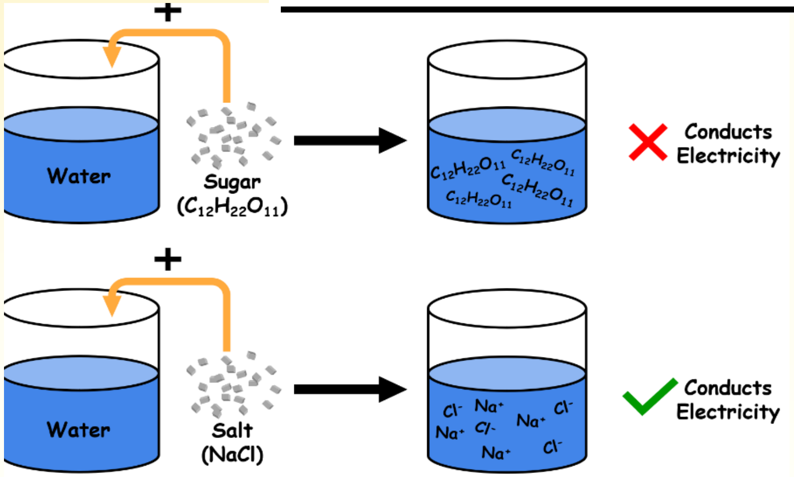
➔ A conductor is a material that contains mobile charges. These charges begin to move as soon as they are in an electrostatic field.

★ examples: ▪ metal: → charge carriers = free electrons

▪ Ionized gas: →charge carriers = ions



▪ Electrolytes :→ charge carriers = ions



I- CONDUCTOR IN ELECTROSTATIC EQUILIBRIUM

➔ A conductor is in equilibrium when all the free charges it contains are at rest.

★ . Free charge: barycenter of a set of charge carriers.

2- Properties of a conductor in equilibrium

➔ The electrostatic field is zero inside any conductor in equilibrium:

$\rightarrow$ Free charges are at rest $\vec{F} = \vec{0} \Leftrightarrow \vec{E} = \vec{0}$

➔ The potential is constant inside and on the surface of a conductor in equilibrium.

$$dV = -\vec{E}.\vec{dl} = 0$$

$\rightarrow \vec{E} = \vec{0} \Rightarrow V = C^{te}$ inside

By continuity, $V = C^{te}$ at the surface

★ The surface of a conductor in equilibrium is an equipotential.

★ The field lines to the surface for a charged conductor are perpendicular

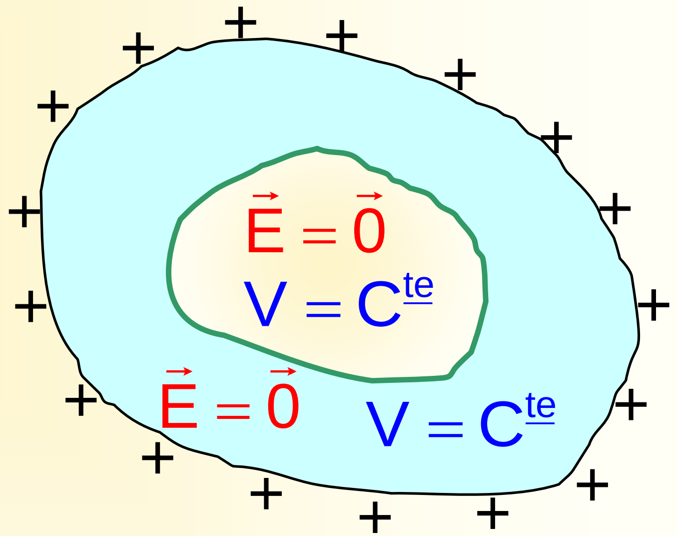
➔ If the conductor in equilibrium is charged, this charge can only exist on its surface.

$$\oiint_{S_G} \vec{E}_{int}\,\vec{n}\,dS_G = \frac{1}{\varepsilon_0}\sum Q_{int} \qquad \vec{E}_{int} = \vec{0} \rightarrow \boxed{\sum Q_{int} = 0}$$

$$(Q^{+}_{int} = Q^{-}_{int})$$

➔ **Case of a charged hollow conductor:**

$\vec{E} = \vec{0} \quad \Rightarrow \quad V = C^{te}$ everywhere



$\rightarrow$ The inner surface is an equipotential.

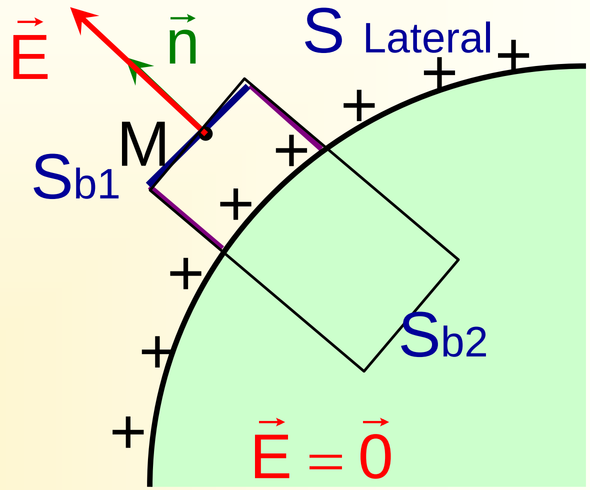
$\rightarrow$ There can be no charge on the inner surface of the cavity.

➔ **Applications:**

★ Charging a conductor, initially neutral, by contact.

3- Field near a conductor

a- Coulomb's theorem :



➔ Hence, the field near a conductor in equilibrium presents a discontinuity across the surface.

- S close surface S_{b1}+ S_{b1}+$S_{lateral}$
- M infinitely close to the surface of the conductor.
- $\vec{n}$ normal to the surface in M.

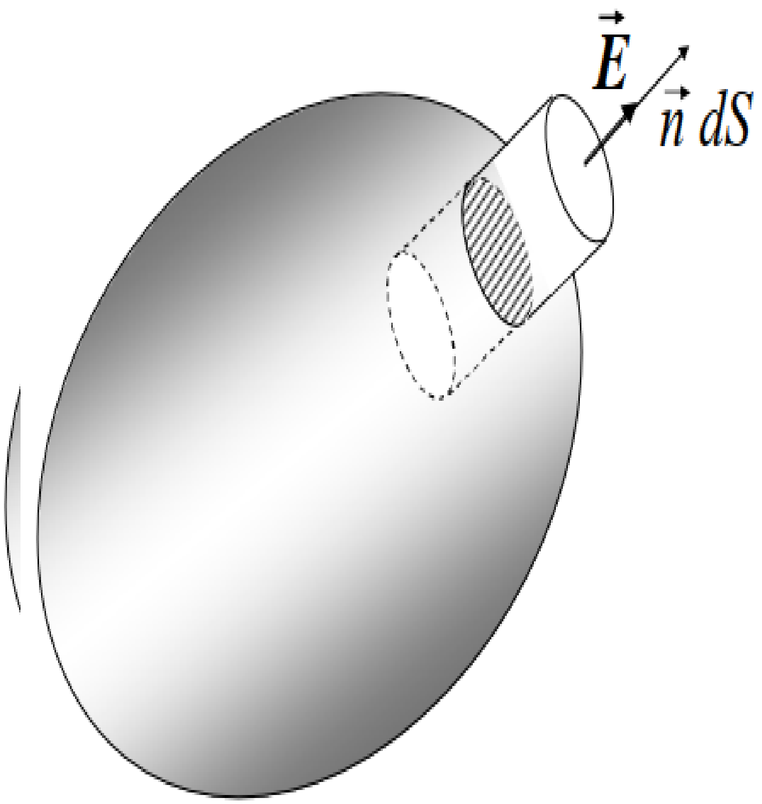
$$\phi = \oiint \vec{E}.\vec{dS} = \frac{Q_{int}}{\varepsilon_0}$$

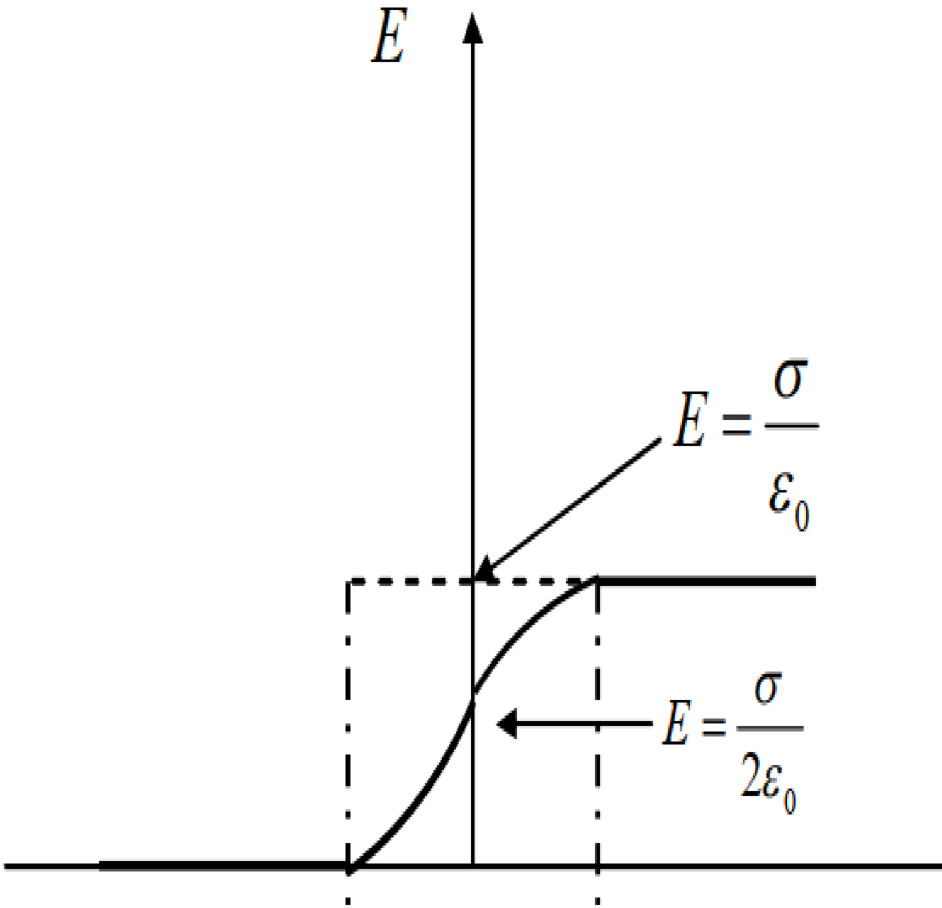
$$\Phi = E\,S_{b1} = \frac{\sigma S_{b1}}{\varepsilon_0}$$

hence $$\vec{E} = \frac{\sigma}{\varepsilon_0} \cdot \vec{n}$$

Field in the vicinity of a conductor in equilibrium

$$E = \frac{\sigma}{\varepsilon_0}$$ or vectorially $$\vec{E} = \frac{\sigma}{\varepsilon_0}\vec{n}$$

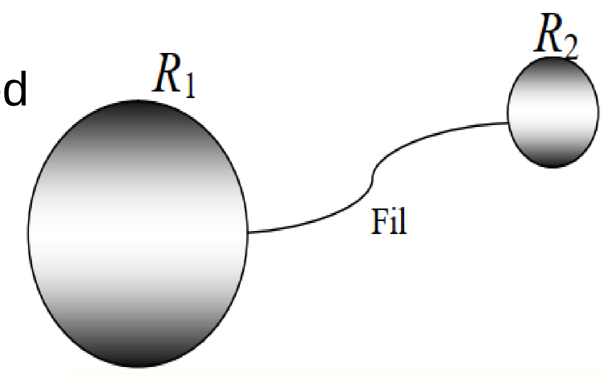




interior Surface Layer exterior

b- "Power of Points"

This phenomenon can be explained by considering two conducting spheres with radii R_1 and R_2 ($R_1 > R_2$), connected by a long thin conducting wire. Consequently, the two spheres are at the same potential; and since they are very far from each other, we can write:



$$V1 = V2 \Leftrightarrow \frac{1}{4\pi\varepsilon_0}\iint_{S_1}\frac{\sigma_1 dS}{R_1} = \frac{1}{4\pi\varepsilon_0}\iint_{S_2}\frac{\sigma_2 dS}{R_2}$$

$$\iint dS = S = 4\pi R^2$$

$$\Leftrightarrow \quad \frac{\sigma_1 R_1}{\varepsilon_0} = \frac{\sigma_2 R_2}{\varepsilon_0}$$

$$\Leftrightarrow \quad \frac{\sigma_1}{\sigma_2} = \frac{R_2}{R_1} \Longrightarrow R_1 \gg R_2 \Longrightarrow \sigma_2 \gg \sigma_1 \Longrightarrow E_2 \gg E_1$$

This last equation shows that the sphere with the smallest radius carries the greatest charge density, therefore a stronger field.

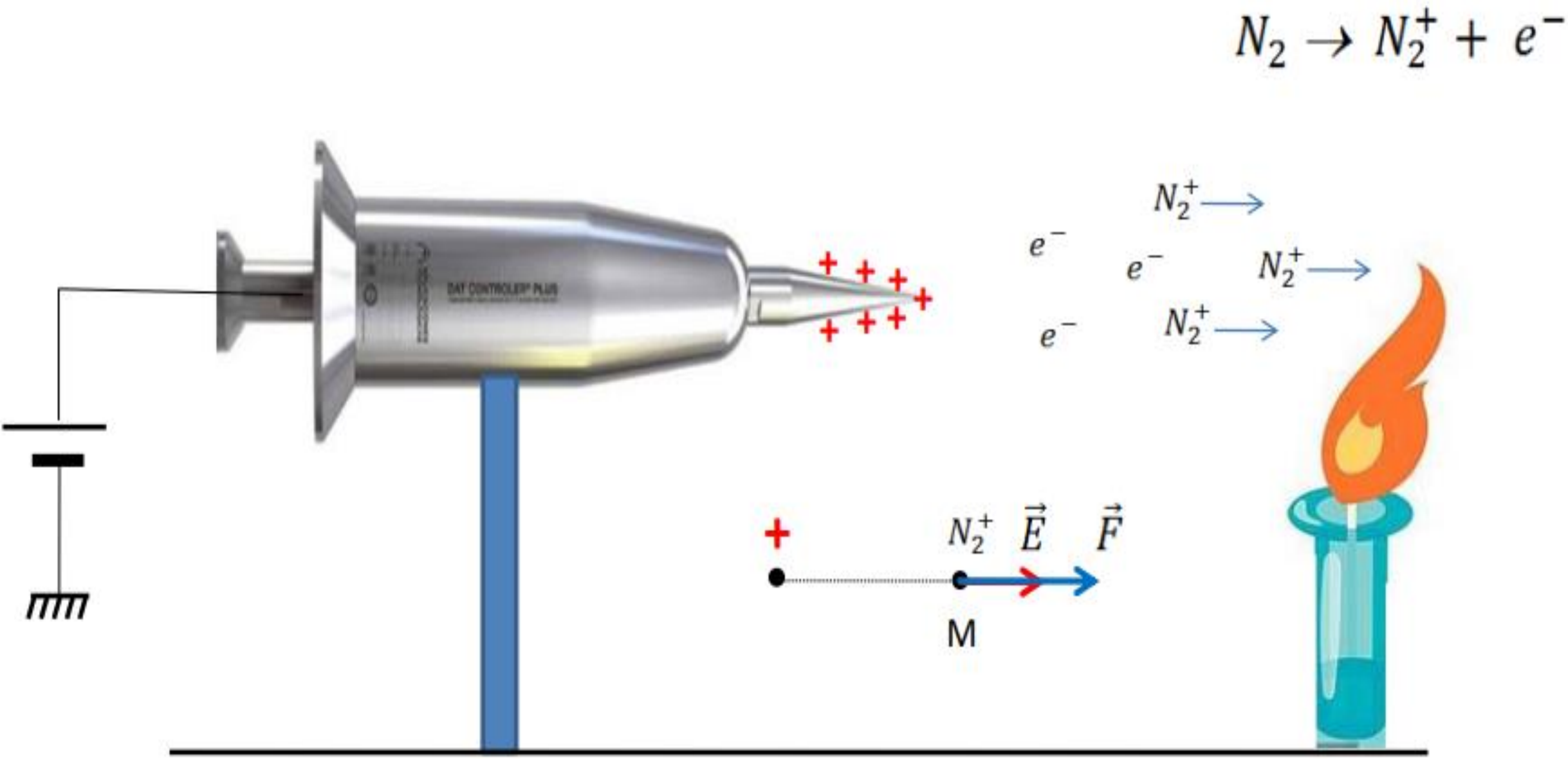
Applications.

Peak power is used to facilitate electrical discharge; this is the function of lightning rods installed on buildings to protect them from lightning.

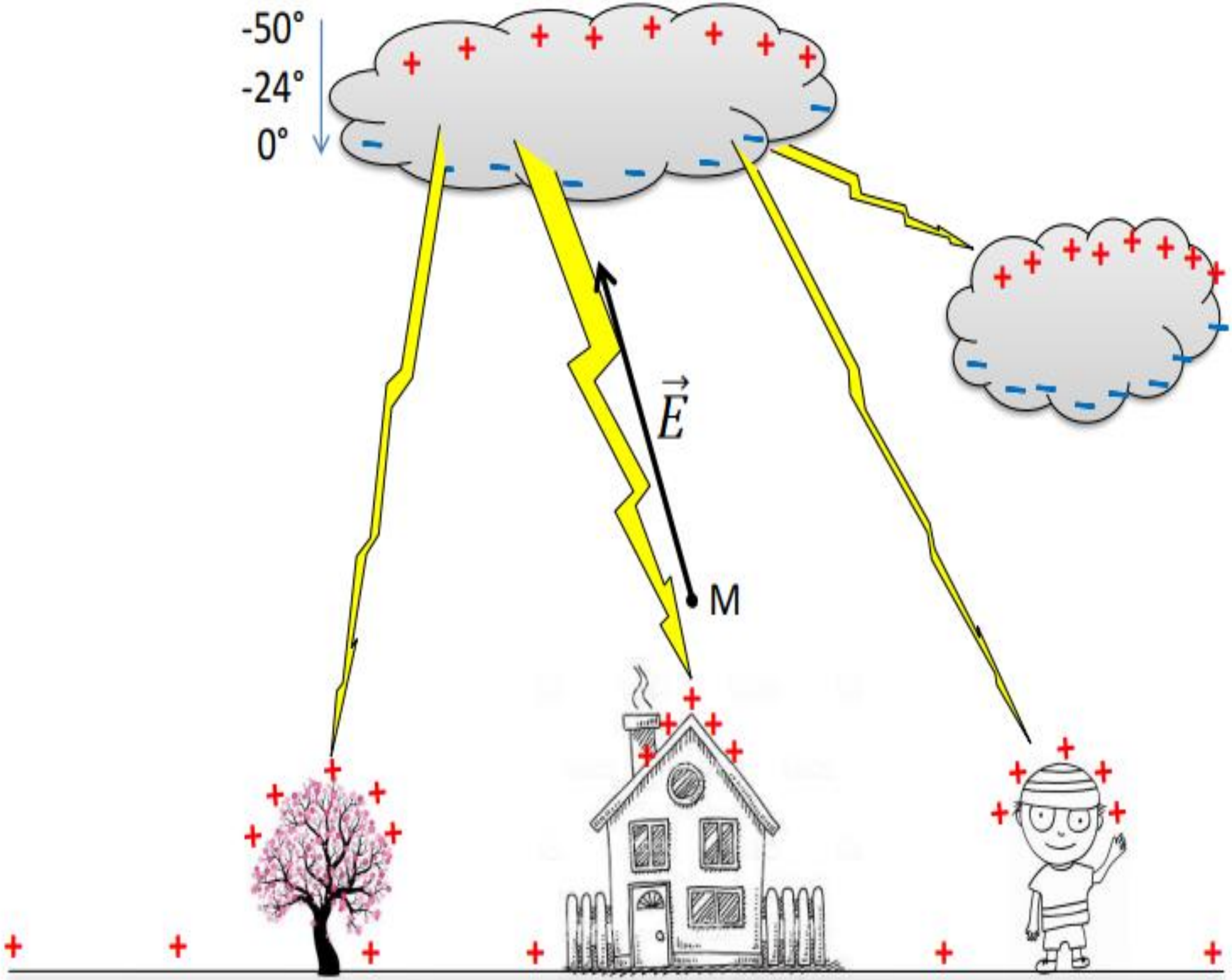
Lightning is a phenomenon of electrical discharge that occurs during a thunderstorm between two clouds charged with static electricity, or between an electrically charged cloud and the Earth, which is an electrical conductor.

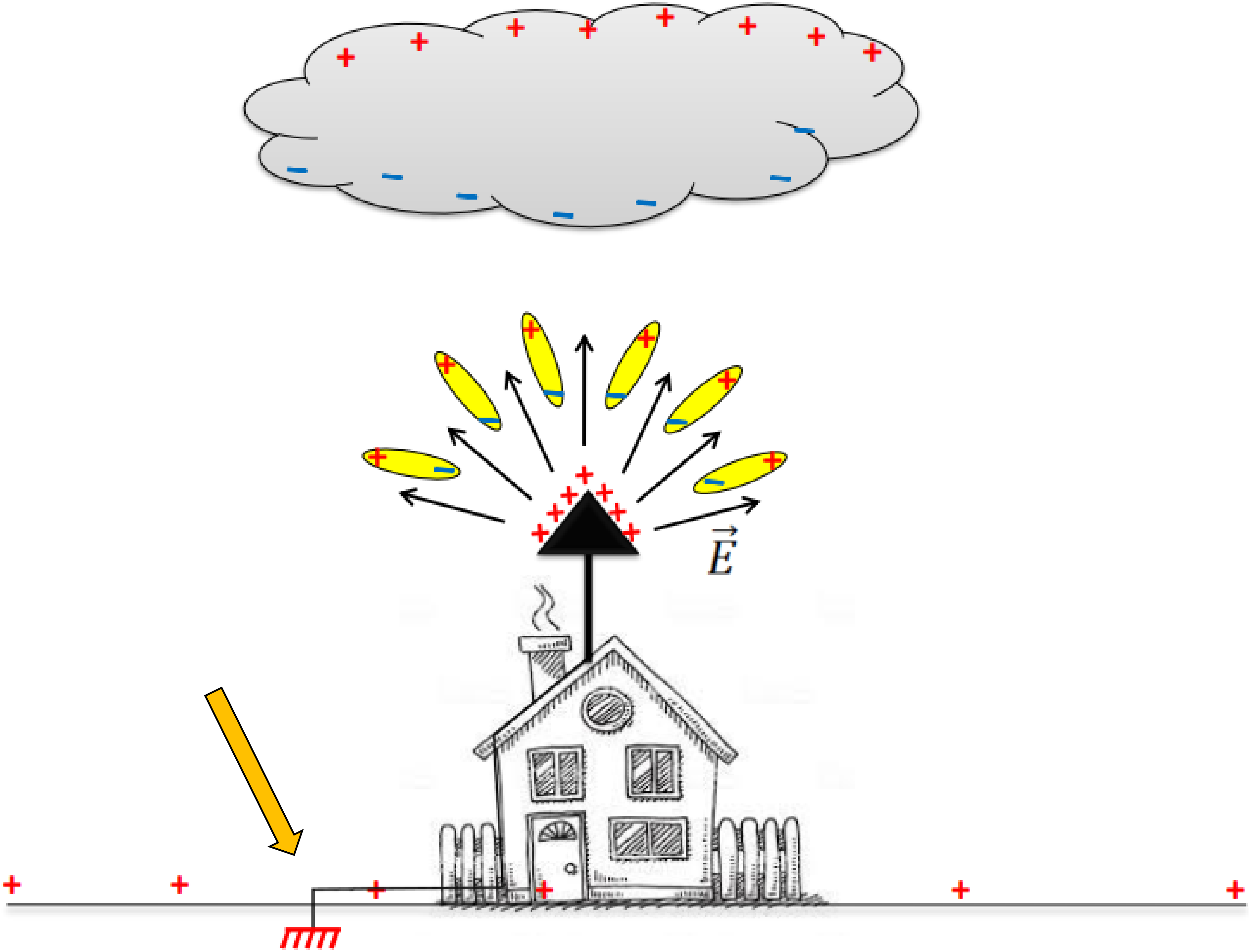
•During a thunderstorm, the components of a cloud—rain droplets, hailstones, and ice particles—collide at very high speeds and become electrified through triboelectricity. The discharge occurs when the potential difference between the cloud and the Earth, for example, exceeds a certain **threshold** (several million volts). Lightning is accompanied by a luminous phenomenon (the flash), and a detonation (the thunder).

Candle experiment: electric wind



Near the tip, the electric field is so intense that the air becomes ionized. The ions with the same charge as the tip are repelled. This results in a movement of air, an "electric wind," which is capable of blowing out the flame of a candle placed near the tip."

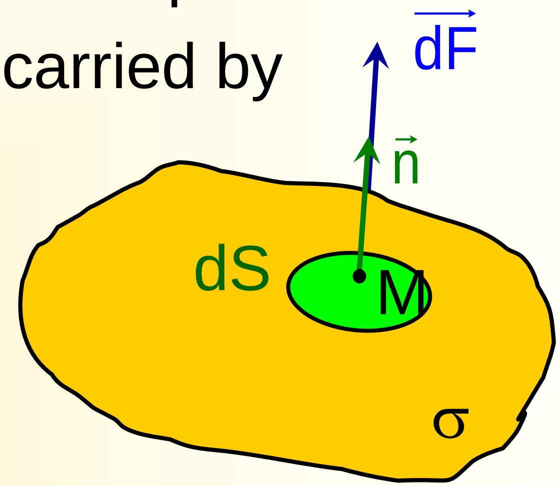

-50°
-24°
0°
$\vec{E}$
M


$\vec{E}$

c- Electrostatic Pressure

Let us take a surface element dS of a conductor in equilibrium.

Let's find the force dF applied to the charge dq carried by dS:



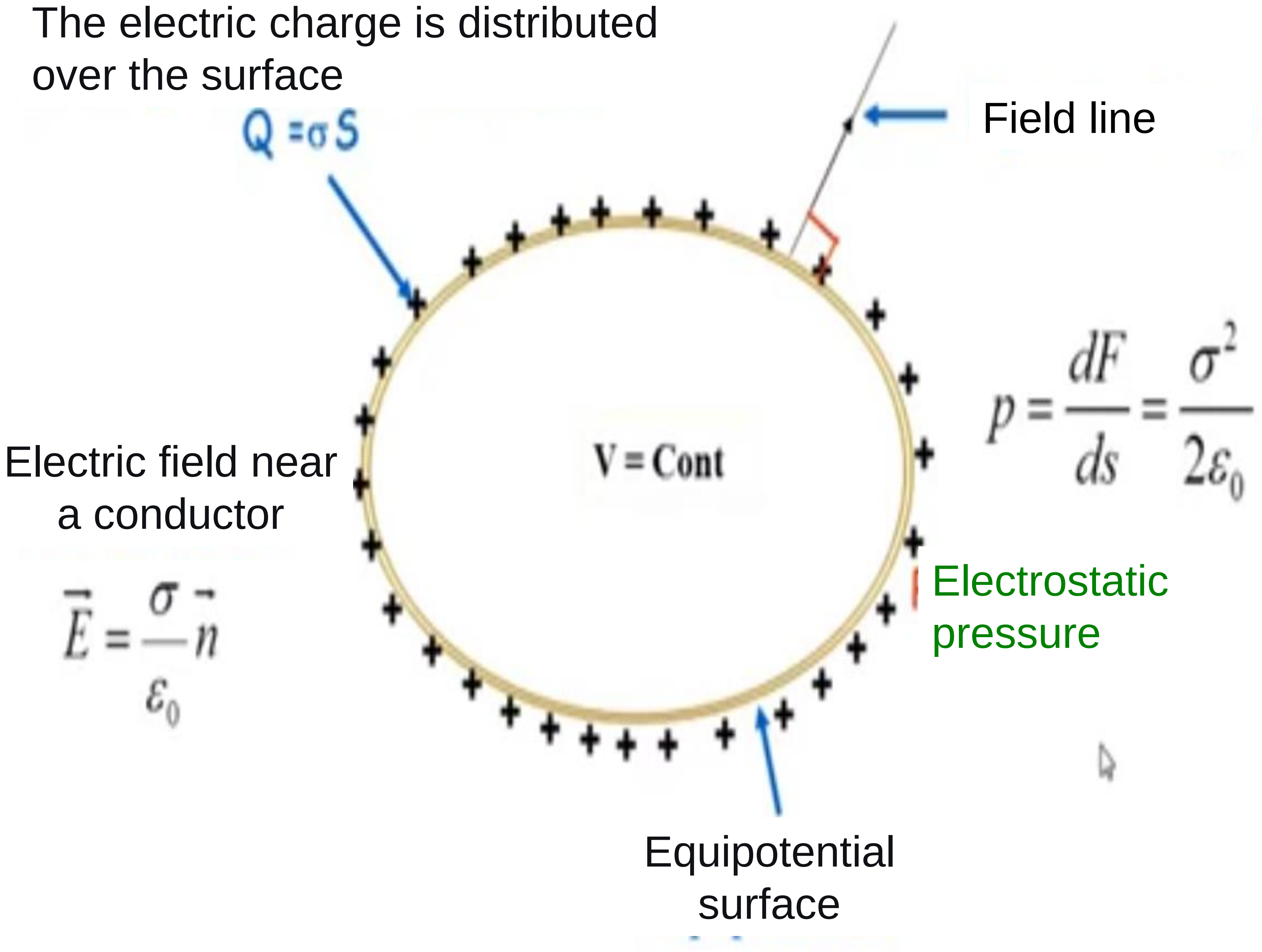
$$\vec{E}(M) = \frac{\sigma}{\varepsilon_0}\vec{n}$$

$$\overrightarrow{dF} = dq \cdot \vec{E}_2 = \frac{\sigma^2}{2\varepsilon_0} \cdot dS \cdot \vec{n}$$

➔ By definition, $p = \frac{dF}{dS} = \frac{\sigma^2}{2\varepsilon_0}$ is the electrostatic pressure

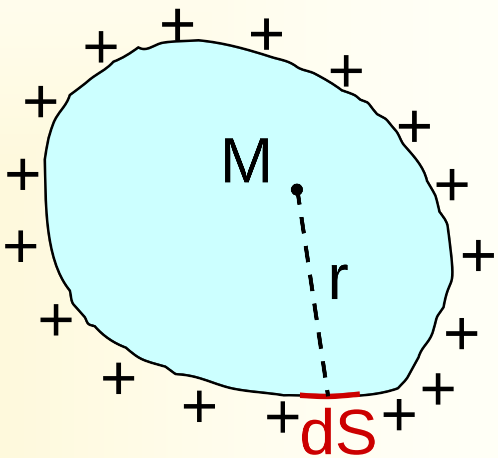
★ $\overrightarrow{dF}$ always directed following $\vec{n}$

★ If σ Very large, the charges leave the conductor
$\Rightarrow$ emission by field effect


The electric charge is distributed over the surface
$Q = \sigma S$
Field line
Electric field near a conductor
$\vec{E} = \frac{\sigma}{\varepsilon_0}\vec{n}$
V = Cont
$p = \frac{dF}{ds} = \frac{\sigma^2}{2\varepsilon_0}$
Electrostatic pressure
Equipotential surface

4- Capacitance of a conductor in equilibrium

For a point M on an equilibrium conductor with surface S and charge density σ, the potential is given by:



$$V(M) = \frac{1}{4\pi\varepsilon_0}\iint_S \frac{\sigma \cdot ds}{r}$$

The total charge is $Q = \iint_S \sigma \cdot dS$

If $\sigma' \rightarrow k\sigma$ then: $\begin{cases} V'(M) = k \cdot V(M) \\ Q' = k \cdot Q \end{cases} \Rightarrow \frac{Q}{V} = \frac{Q'}{V'} = C^{te}$

- Hence, the capacitance of an isolated conductor is defined as:

$$\frac{Q}{V} = C$$

★ C >0

★ Unity: Farad (coulomb.volt^{-1})

★ Example: $C_{earth} = 4\pi\varepsilon_0 R = 710\ \mu F$

Consider an isolated conductor carrying a charge Q, which creates a potential V.

$$Q = C.V$$

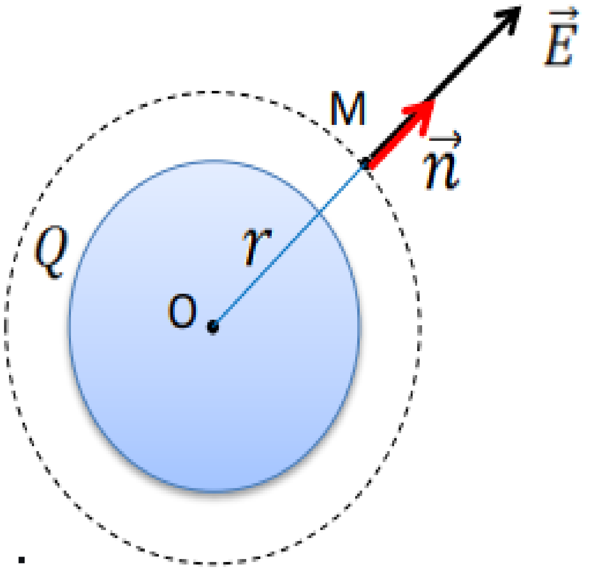
Self-capacitance of conductor

$$C = \frac{Q}{V}$$

$$[C] = \frac{[Q]}{[V]} = \frac{Coulomb(C)}{Volt(V)} = Farad\ (F)$$

$$1mF = 10^{-3}F$$
$$1\mu F = 10^{-6}F$$
$$1nF = 10^{-9}F$$
$$1pF = 10^{-12}F$$
$$1fF = 10^{-15}F$$

Example: Calculate the self-capacitance of a spherical conductor of radius R.



The field :

$$\oiint_{S_G} \vec{E}.\vec{n}\, dS_G = \frac{1}{\varepsilon_0}\sum Q_{int}$$

$$E.S_G = \frac{Q}{\varepsilon_0} \rightarrow E = \frac{Q}{4\pi\varepsilon_0 r^2}$$

Potential :

$$dV = -\vec{E}\,.\,\overrightarrow{dr}$$

$$V(r) = -\int \vec{E}\,.\,\overrightarrow{dr} + \lambda = \frac{Q}{4\pi\varepsilon_0 r} + \lambda$$

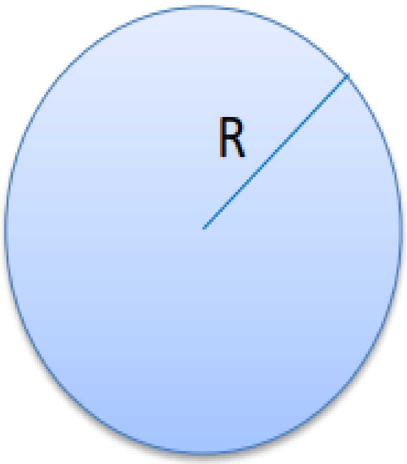
$$V(\infty) = 0 \rightarrow \lambda = 0$$

$$V = V(R) = \frac{Q}{4\pi\varepsilon_0 R}$$

$$C = \frac{Q}{V} = 4\pi\varepsilon_0 R$$

The capacitance depends on the geometry and the material of the conductor.

Calculation of the Earth's capacitance



$$K = \frac{1}{4\pi\varepsilon_0} = 9\times10^9\ S.I$$

$$C = \frac{R}{K} = 0.71\ mF$$

$$4\pi\varepsilon_0 R$$

5. Electrostatic Energy of a Charged Isolated Conductor

The electrostatic energy of an isolated conductor is the work required to charge with a charge Q.

$$dE_p = V\,dq$$

$$E_p = \int_0^Q V\,dq = \frac{1}{C}\int_0^Q q\,dq$$

For $Q = C\,V$

$$E_p = \frac{1}{2C}Q^2 = \frac{1}{2}CV^2 = \frac{1}{2}QV$$

In the case of n conductors in equilibrium, we have:

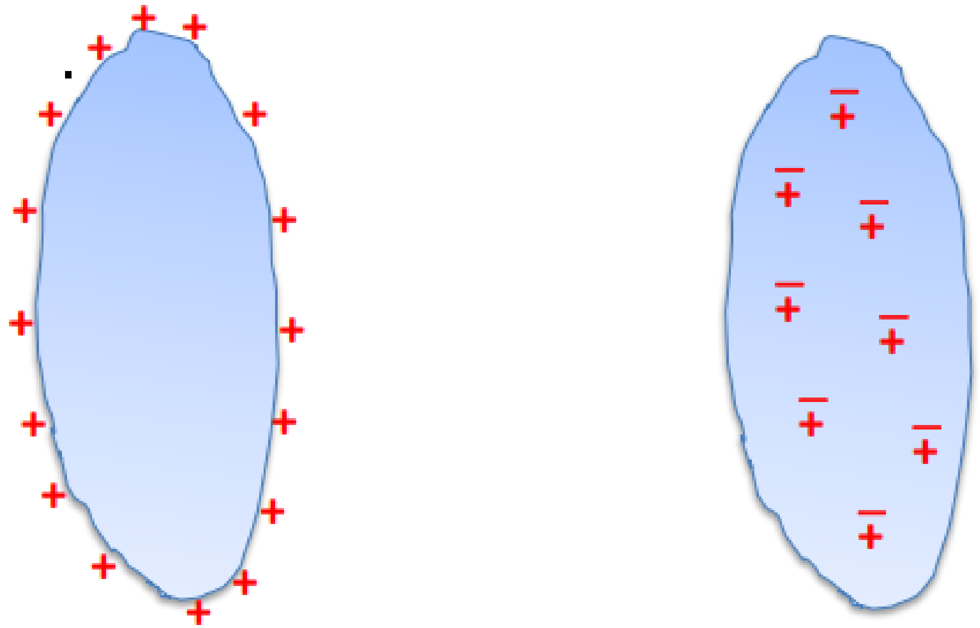
$$E_p = \frac{1}{2}\sum_{i=1}^{n} Q_i V_i$$

6. Electrostatic Influence Phenomena

a) Definition

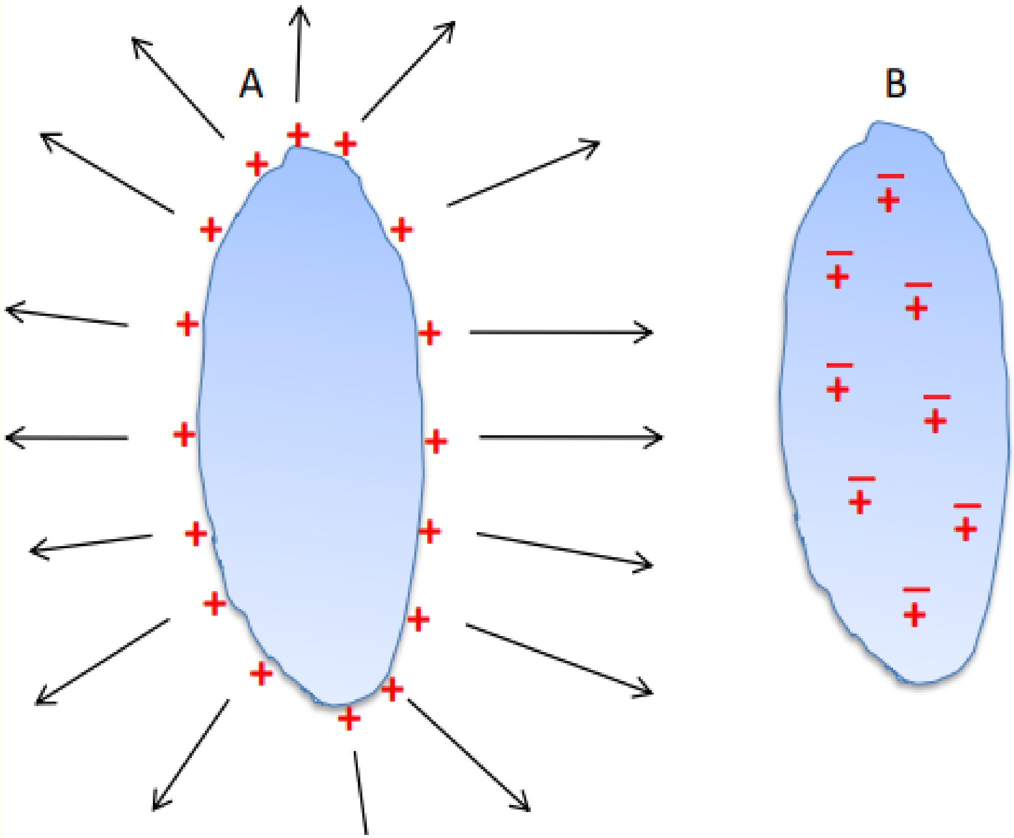
Electrostatic influence

Change in the surface charge density of a conductor under the influence of another charged conductor.



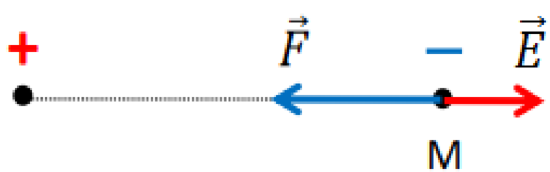
Charged (influencing) Neutral (influenced)

b) partial influence

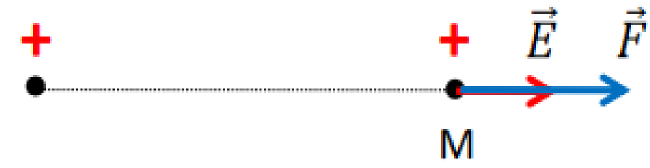


Only some of the field lines leaving A arrive at B.

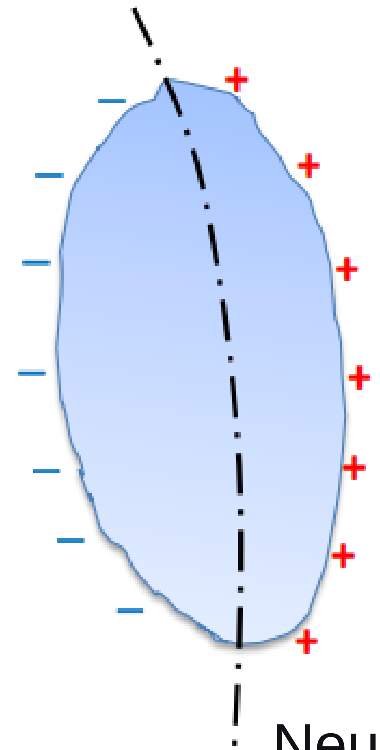
$$\vec{F} = -e\,\vec{E}$$



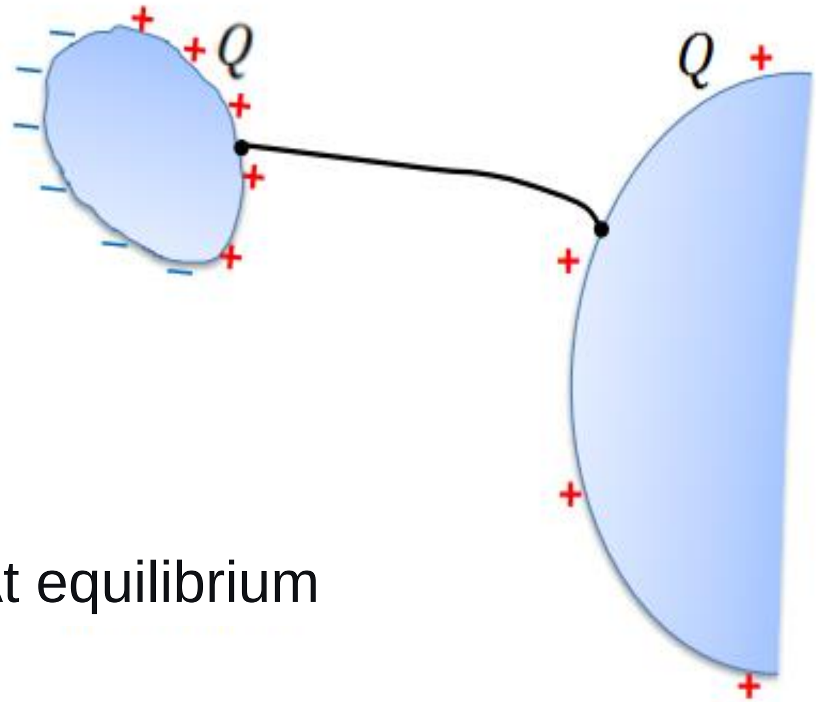
$$\vec{F} = e\,\vec{E}$$



Result:

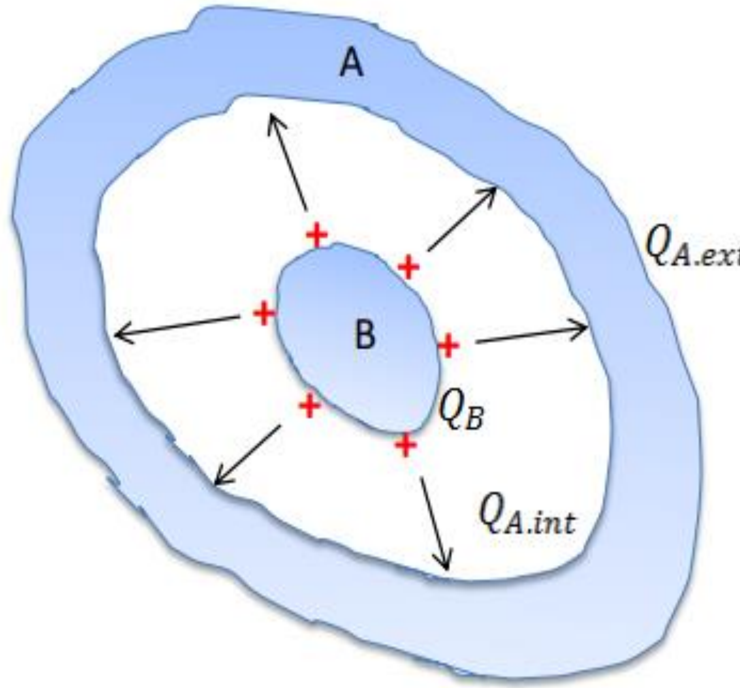


Discharging a polarized conductor



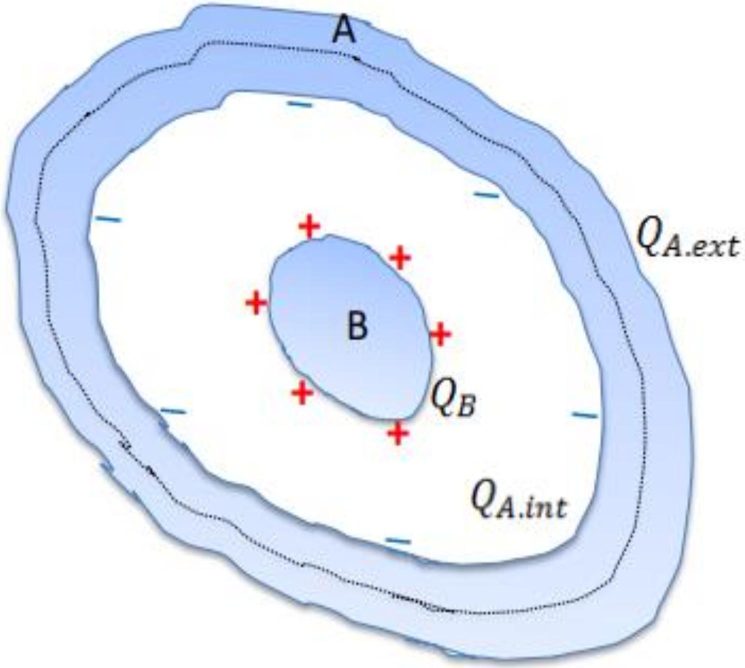
At equilibrium

b) Total influence



All of the field lines leaving A arrive at B.

$Q_{A.int}$, $Q_{A.ext}$?



- (A) initially neutral $(Q_A = 0)$

$$\oiint_{S_G} \vec{E}_{int}.\vec{n}\, dS_G = \frac{1}{\varepsilon_0}\sum Q_{int}^{SG}$$

$$\vec{E}_{int} = \vec{0} \rightarrow \sum Q_{int}^{SG} = 0$$

$$Q_{A.int} + Q_B = 0 \rightarrow \boxed{Q_{A.int} = -Q_B}$$

The total charge of A

$$Q'_A = Q_{A.int} + Q_{A.ext}$$

According to the conservation law for A.

$$Q'_A = Q_A = 0$$

$$Q_{A.int} + Q_{A.ext} = 0$$

$$\boxed{Q_{A.ext} = -Q_{A.int} = Q_B}$$

- (A) Initially charged $(Q_A = Q_0)$

As before

$$\boxed{Q_{A.int} = -Q_B}$$

According to the conservation

$$Q'_A = Q_A = Q_0$$

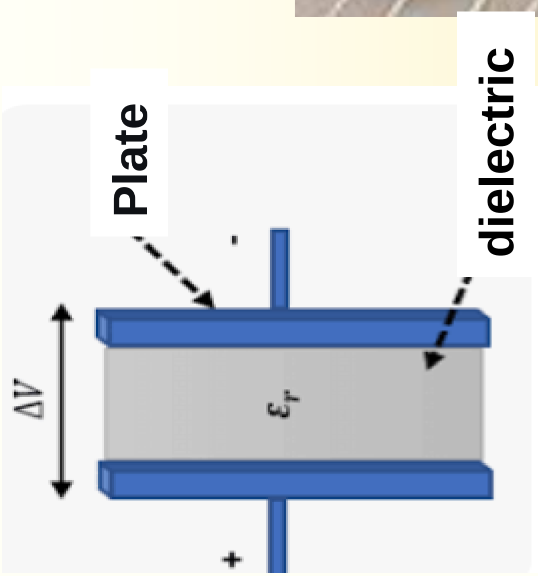
$$\boxed{Q_{A.ext} = Q_0 + Q_B}$$

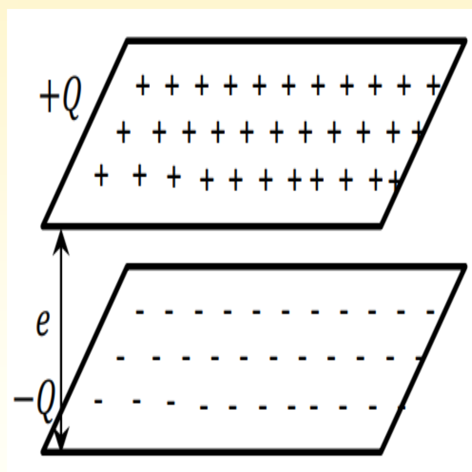
The capacitors

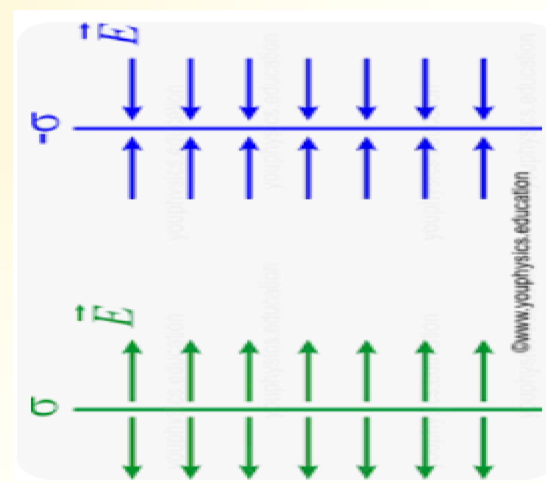
Definition : A set of two conductors in total influence

7. The capacitors

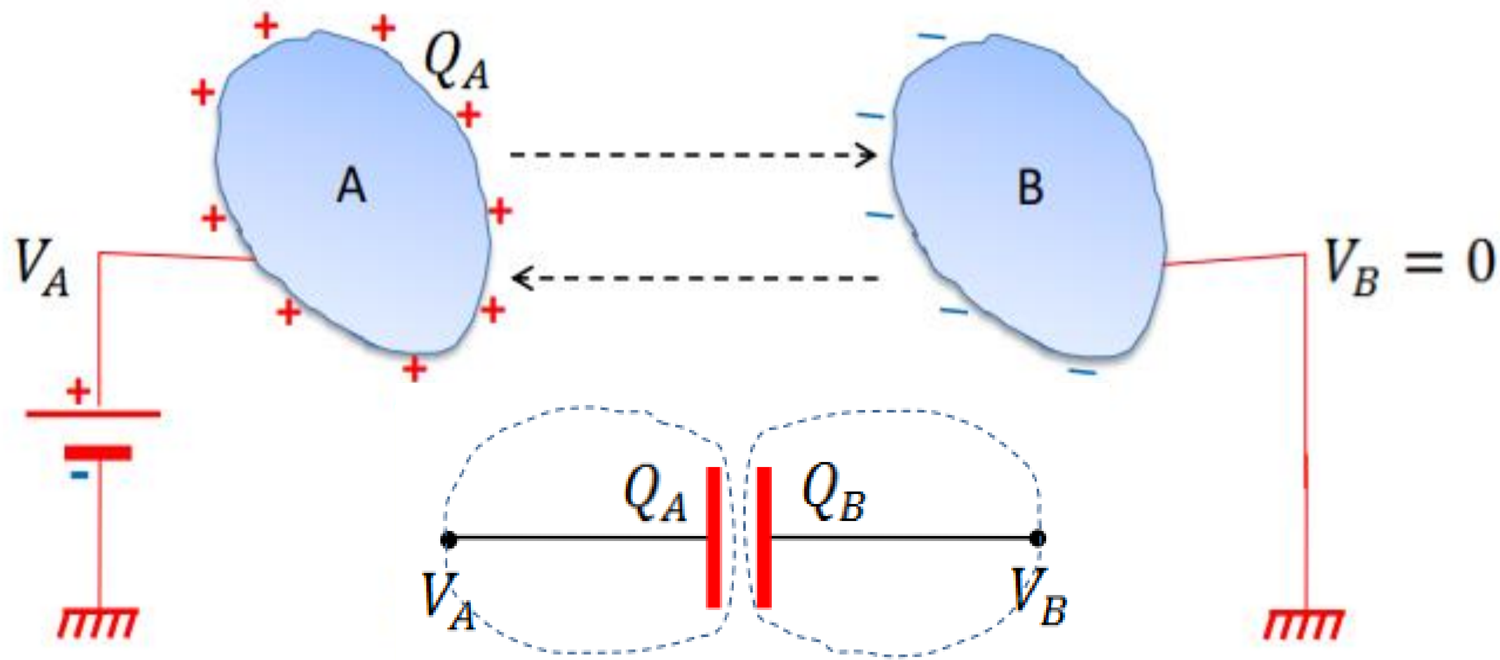
A capacitor is a device used to store electrical energy. It is widely used in electronics and electrical engineering.







Consider a conductor *A* with capacitance *C*, maintained at a potential V_A ($V_A > 0$) It carries a charge Q_A.



The charge of the capacitor $Q = |Q_A| = |Q_B|$

The capacitance of the capacitor

$$C = \frac{Q}{|V_A - V_B|}$$

2- Calculation of Capacitance

a- General method

① → Calculate the electric field $\vec{E}$ between the plates (Gauss's theorem).

② → Calculation of the circulation of $\vec{E}$ between the plates.

$$V_1 - V_2 = \int_{A_1}^{A_2} \vec{E} \cdot \overrightarrow{dl}$$

③ → Calcul de C.

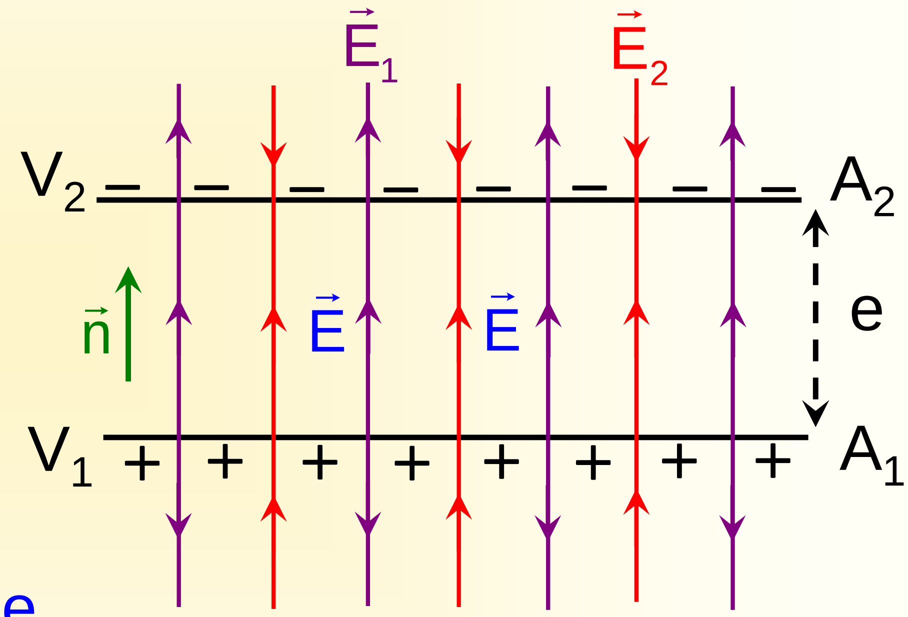
$$Q = \iint_S \sigma \cdot dS \qquad \text{et} \qquad C = \frac{Q}{V_1 - V_2}$$

b- Parallel-Plate Capacitor

→ Plates A_1 and A_2: parallel plates with surface area S, separated by a distance e (where S >> e), held at potentials V1 and V2.

→ Uniform charge densities: $+\sigma$ on A_1 et $-\sigma$ on A_2.

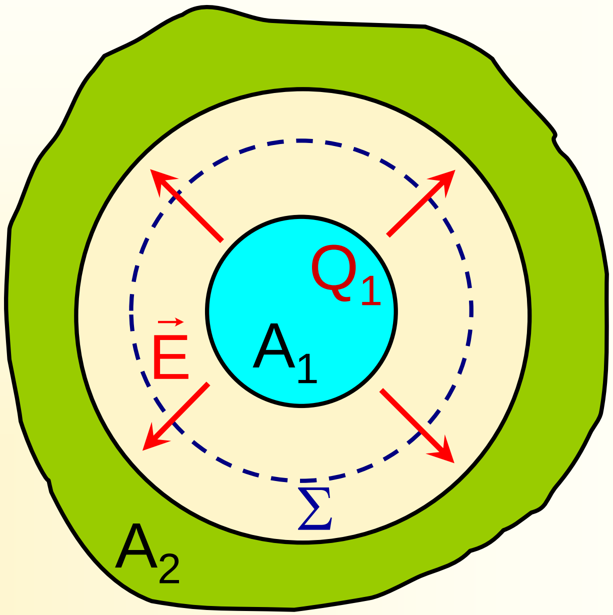
① $$\left.\begin{aligned}\vec{E}_1 &= \pm\frac{\sigma}{2\varepsilon_0}\vec{n} \\ \vec{E}_2 &= \pm\frac{\sigma}{2\varepsilon_0}\vec{n}\end{aligned}\right\} \Rightarrow \vec{E} = \frac{\sigma}{\varepsilon_0}\vec{n}$$



② $$V_1 - V_2 = \int_{A_1}^{A_2} \vec{E}\cdot\vec{dl} = E\cdot e = \frac{\sigma\cdot e}{\varepsilon_0}$$

③ $$Q = \sigma\cdot S \quad \Rightarrow \quad C = \frac{\varepsilon_0 S}{e}$$

c- spherical capacitor



→ Plate A_1: sphere of radius R_1.

→ Plate A_2: sphere of radius R_2.

① Σ Gauss surface of radius r ⇒

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{Q_1}{r^2}\vec{u}_r$$

② Circulation of $\vec{E}$ between A_1 and A_2:

$$\int_{A_1}^{A_2}\vec{E}\cdot\overrightarrow{dr} = V_1 - V_2 = \frac{Q_1}{4\pi\varepsilon_0}\int_{R_1}^{R_2}\frac{dr}{r^2}$$

③ $$C = \frac{Q_1}{V_1 - V_2} = 4\pi\varepsilon_0\left(\frac{R_1R_2}{R_2 - R_1}\right)$$

1°/ Field calculation

$$\oiint_{S_G} \vec{E}.\vec{n}\, dS_G = \frac{1}{\varepsilon_0} \sum Q_{int}$$

$$E.S_G = \frac{Q}{\varepsilon_0} \rightarrow E = \frac{Q}{4\pi\varepsilon_0 r^2}$$

2°/ circulation calculation

$$\int_{V_A}^{V_B} dV = - \int_{R_1}^{R_2} \vec{E}.\vec{n}.\overrightarrow{dr}$$

$$V_A - V_B = \frac{Q}{4\pi\varepsilon_0} \int_{R_1}^{R_2} \frac{dr}{r^2}$$

$$V_A - V_B = \frac{Q}{4\pi\varepsilon_0} \int_{R_1}^{R_2} \frac{dr}{r^2}$$

$$V_A - V_B = \frac{Q}{4\pi\varepsilon_0} \left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

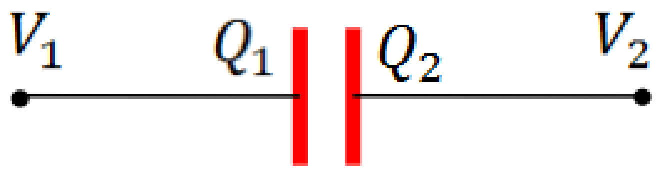
3°/ Capacitance calculation

$$C = \frac{Q}{V_A - V_B} = 4\pi\varepsilon_0 \left(\frac{R_1 R_2}{R_2 - R_1}\right)$$

8. Stored energy in a capacitor

To calculate this energy, we use:

$$E_p = \frac{1}{2}\sum_{i=1}^{n} Q_i V_i$$



$$E_p = \frac{1}{2}(Q_1 V_1 + Q_2 V_2)$$

$$Q_1 = Q = -Q_2$$

$$E_p = \frac{1}{2}(V_1 - V_2)Q$$

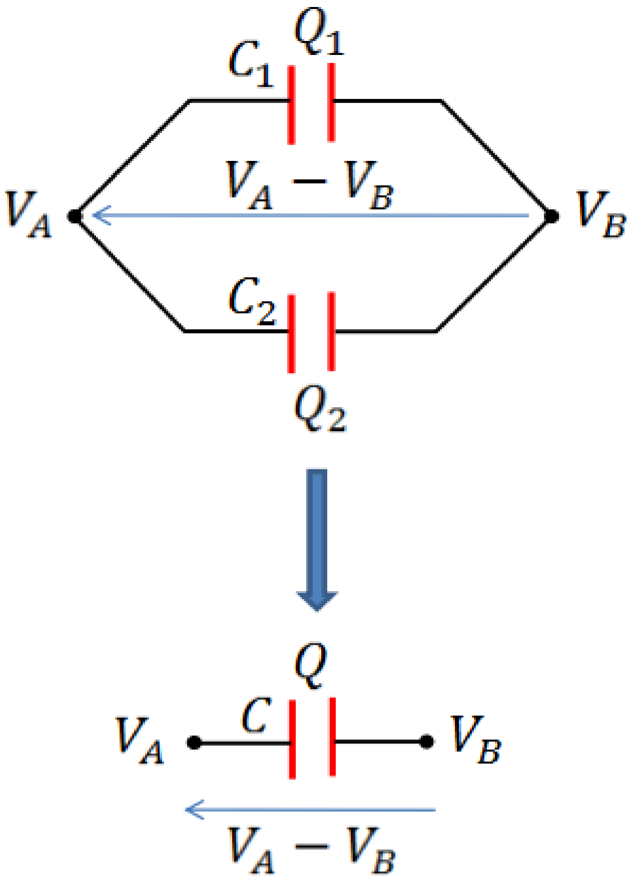
$$Q = C(V_1 - V_2)$$

$$\boxed{E_p = \frac{1}{2}C(V_1 - V_2)^2}$$

9. Capacitor combinations

There are two types of capacitor arrangements: series and parallel.

b) parallel connection



$$Q = Q_1 + Q_2$$

$$C(V_A - V_B)$$

$$C_1(V_A - V_B) + C_2(V_A - V_B)$$

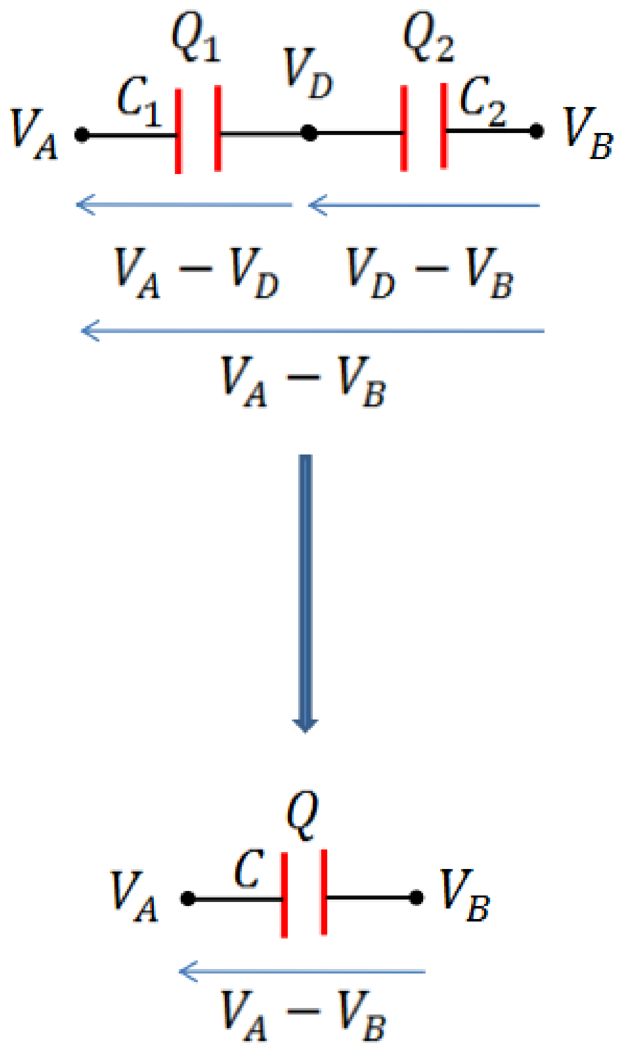
$$V_A \neq V_B$$

$$C = C_1 + C_2$$

For n capacitors:

$$C = \sum_{i=1}^{n} C_i$$

b) series connection



$$V_A - V_B = (V_A - V_D) + (V_D - V_B)$$

$$\frac{Q}{C} \qquad\qquad \frac{Q_1}{C_1} + \frac{Q_2}{C_2}$$

$$Q = Q_1 = Q_2 \neq 0$$

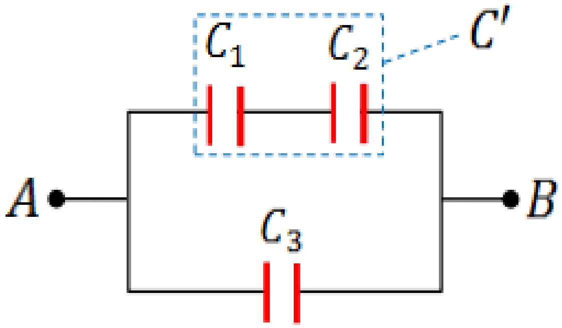
$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} \rightarrow C = \frac{C_1 C_2}{C_1 + C_2}$$

For n capacitors:

$$\boxed{\frac{1}{C} = \sum_{i=1}^{n} \frac{1}{C_i}}$$

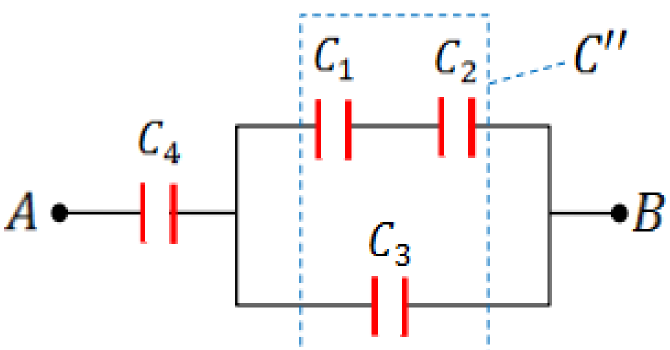
Example 1: For each case, calculate the equivalent capacitance between points A and B.

(a)



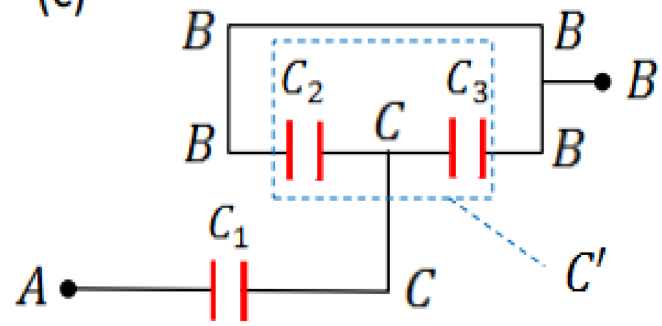
$$C' = \frac{C_1 C_2}{C_1 + C_2} \qquad C_{AB} = \frac{C_1 C_2}{C_1 + C_2} + C_3$$

(b)



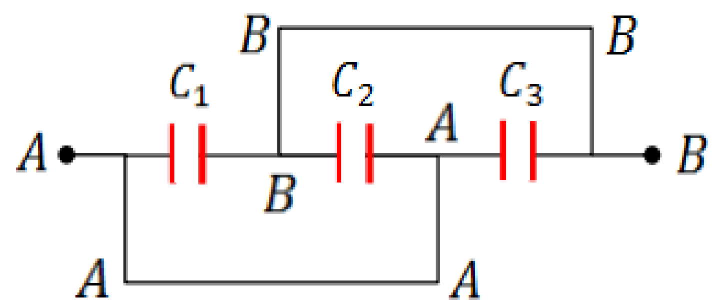
$$C'' = \frac{C_1 C_2}{C_1 + C_2} + C_3 \qquad C_{AB} = \frac{C_4 C''}{C_4 + C''}$$

(c)



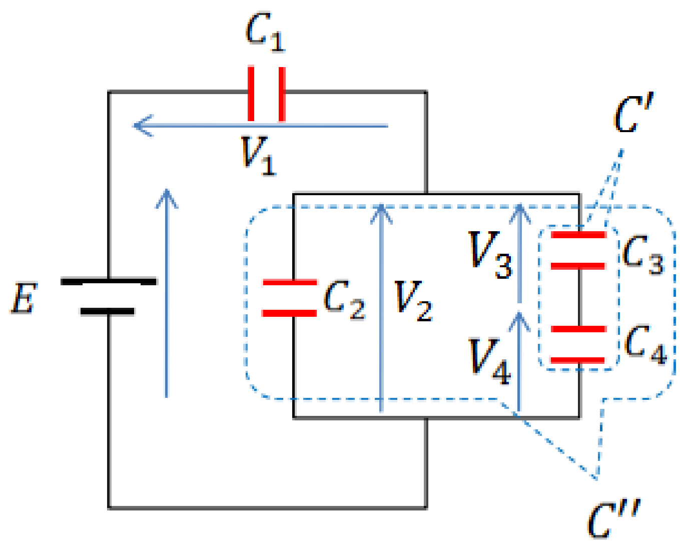
$$C' = C_2 + C_3 \qquad C_{AB} = \frac{C_1 C'}{C_1 + C'}$$

(d)



$$C_{AB} = C_1 + C_2 + C_3$$

Example 2: Calculate the charge and potential difference across each capacitor.



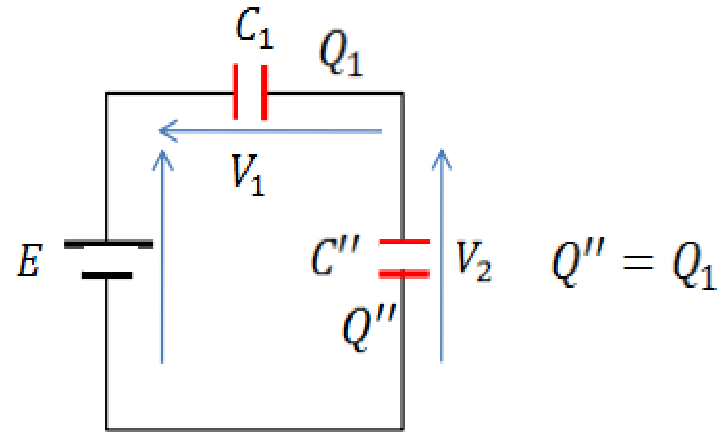
$$V_i = \frac{Q_i}{C_i} \qquad V_2 = V_3 + V_4$$

$$E = V_1 + V_2 \qquad Q_3 = Q_4$$

$$C' = \frac{C_3 C_4}{C_3 + C_4}$$

$$C'' = C_2 + C' = C_2 + \frac{C_3 C_4}{C_3 + C_4}$$

$$E = \frac{Q_1}{C_1} + \frac{Q''}{C''} = Q_1 \left(\frac{C_1 + C''}{C_1 C''} \right)$$



$$Q_1 = \frac{C_1 C''}{C_1 + C''} E \;\rightarrow\; V_1 = \frac{C''}{C_1 + C''} E$$

$$V_2 = E - V_1$$

$$V_2 = \frac{C_1}{C_1 + C''} E \;\rightarrow\; Q_2 = \frac{C_1 C_2}{C_1 + C''} E$$

$$V_2 = \frac{Q_3}{C_3} + \frac{Q_4}{C_4} = Q_3 \left(\frac{C_3 + C_4}{C_3 C_4} \right)$$

$$Q_3 = Q_4 = \frac{C_3 C_4}{C_3 + C_4} V_2 = C' \, V_2$$

$$V_3 = \frac{C_4}{C_3 + C_4} V_2 \qquad V_4 = \frac{C_3}{C_3 + C_4} V_2$$

N, A,: $E = 33\ V\ ,\ C_i = i\ \mu F$

$$C' = \frac{C_3 C_4}{C_3 + C_4} = \frac{12}{7}\mu F$$

$$C'' = C_2 + C' = \frac{26}{7}\mu F$$

$$Q_1 = \frac{C_1 C''}{C_1 + C''}E \rightarrow \boxed{Q_1 = 26\ \mu C} \rightarrow V_1 = \frac{Q_1}{C_1} \rightarrow \boxed{V_1 = 26\ V}$$

$$V_2 = E - V_1 \rightarrow \boxed{V_2 = 7\ V}$$

$$Q_2 = C_2 V_2 \rightarrow \boxed{Q_2 = 14\ \mu C}$$

$$Q_3 = Q_4 = \frac{C_3 C_4}{C_3 + C_4}V_2 = C'\ V_2 \rightarrow \boxed{Q_3 = Q_4 = 12\ \mu C}$$

$$V_3 = \frac{Q_3}{C_3} \rightarrow \boxed{V_3 = 4\ V}$$

$$V_4 = \frac{Q_4}{C_4} \rightarrow \boxed{V_4 = 3\ V}$$

The End