Chapter 01: Kinematics of a material point

I.1. Objective:

The aim of kinematics is to describe in qualitative terms the motion of a body without looking at the causes that produce it.

The study of the motion of a body is based on the study of its successive positions relative to a reference frame, as well as its velocity and acceleration and the relationships between these three quantities as a function of time.

I.2. Definitions:

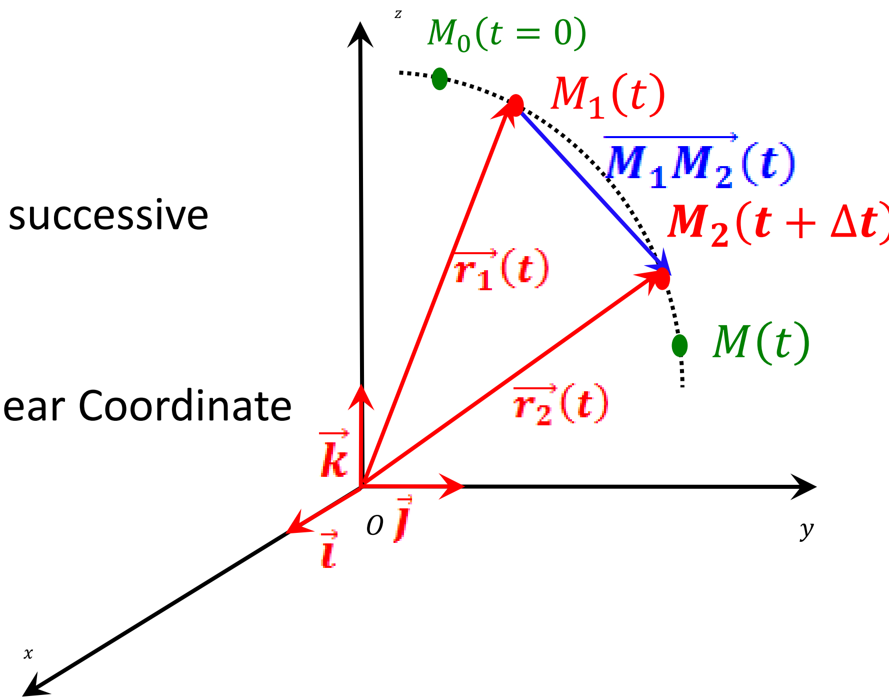
- ***Material point (particle) :*** An object with negligible dimensions on a macroscopic scale, which is assimilated to a geometric point.

In reality, the study of the motion of an object can be described by:

- **Motion around its center of mass.**
- **Motion of its own center of mass**

❑ *Trajectory, Curvilinear Coordinates,* Equations of motion

المعادلة الزمنية للحركة - الاحداثيات المنحنية – المسار



Consider "M" as a moving particle in space

➢ The trajectory "M" is the geometric locus of the successive positions occupied by "M" over time.

➢ The algebraic value $\widehat{M_0M}(t)$ is called the Curvilinear Coordinate

➢ $M_0M(t) = S(t)$: Equation of motion of « M »

➢ $\overrightarrow{OM_1}$ *et* $\overrightarrow{OM_2}$: is a Position Vectors of « M » /« O »

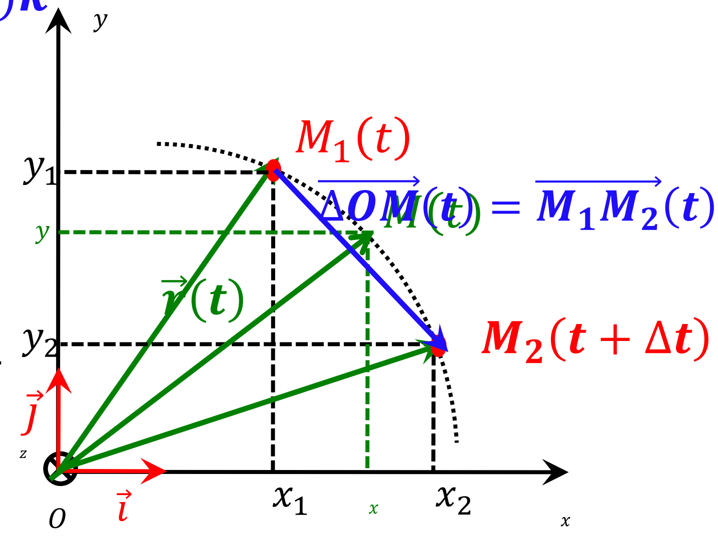
➢ $\overrightarrow{M_1M_2}(t)$: Displacement vector of « M » from position M_1 (t) to position M_2(t+ Δt)

$$\left|\overrightarrow{M_1M_2}(t)\right| = \left|\overrightarrow{\Delta OM}(t)\right| = \left|\overrightarrow{OM_2}(t) - \overrightarrow{OM_1}(t)\right| = \left|\vec{r_2}(t) - \vec{r_1}(t)\right|$$

II. Curvilinear Motion

The motion of "M" is defined by its position vector at each time "t" with:

$$\overrightarrow{OM}(t) = \vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$$



II.1. Velocity :

- **Average Velocity:**

$$\vec{v}_{ave} = \frac{\overrightarrow{\Delta OM}(t)}{\Delta t} = \frac{\overrightarrow{M_1M_2}(t)}{\Delta t} = \frac{\overrightarrow{OM_2}(t+\Delta t) - \overrightarrow{OM_1}(t)}{\Delta t}$$

In Cartesian coordinates:

$$\left.\begin{array}{l}\overrightarrow{OM_1}(t) = x_1\vec{i} + y_1\vec{j} + z_1\vec{k} \\ \overrightarrow{OM_2}(t) = x_2\vec{i} + y_2\vec{j} + z_2\vec{k}\end{array}\right\} \Rightarrow \vec{v}_{ave} = \frac{x_2 - x_1}{\Delta t}\vec{i} + \frac{y_2 - y_1}{\Delta t}\vec{j} + \frac{z_2 - z_1}{\Delta t}\vec{k}$$

$$\vec{v}_{ave} = \frac{\Delta x}{\Delta t}\vec{i} + \frac{\Delta y}{\Delta t}\vec{j} + \frac{\Delta z}{\Delta t}\vec{k}$$

➢ Instantaneous velocity

We obtain it by computing the average velocity for a smaller time interval.

$$\vec{V}_{inst} = \lim_{\Delta t \to 0} \vec{V}_{moy} = \lim_{\Delta t \to 0} \frac{\overrightarrow{\Delta OM}(t)}{\Delta t} = \frac{\overrightarrow{dOM}(t)}{dt} = \frac{\overrightarrow{dr}(t)}{dt}$$

Operationally, the instantaneous velocity is found by observing the moving body at two very close positions separated by the small distance dx and measuring the small time interval dt required to go from one position to the other position.

En coordonnées cartésiennes :

$$\vec{V}_{inst} = \vec{V}(t) = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k}$$

$$\begin{cases} V_x = \dfrac{dx}{dt} \\ V_y = \dfrac{dy}{dt} \\ V_z = \dfrac{dz}{dt} \end{cases}$$

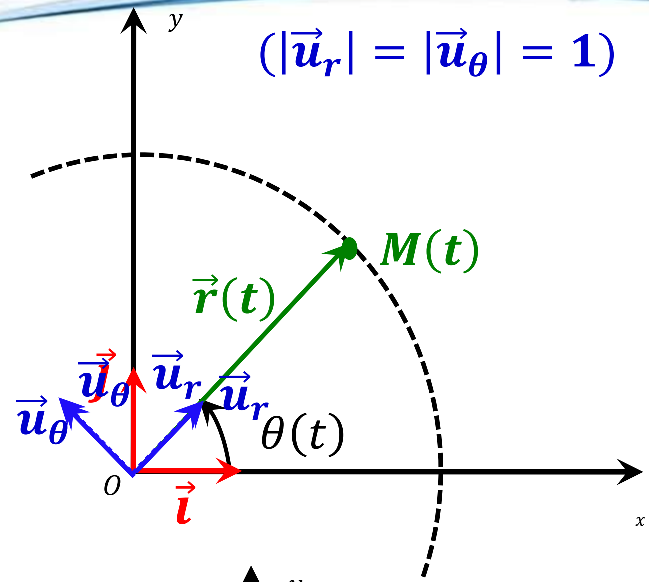
With $|\vec{V}| = V = \sqrt{{V_x}^2 + {V_y}^2 + {V_z}^2}$

In polar coordinates (2d):

$(|\vec{u}_r| = |\vec{u}_\theta| = 1)$

$$\overrightarrow{OM}(t) = \vec{r}(t) = r(t)\vec{u}_r \quad \big(\boldsymbol{r(t), \theta(t)}\big)\text{: Polar coordinates}$$

$$\vec{V}(t) = \frac{d\overrightarrow{OM}(t)}{dt} = \frac{d(r(t)\vec{u}_r)}{dt} = \frac{dr(t)}{dt}\vec{u}_r + r(t)\frac{d\vec{u}_r}{dt}$$

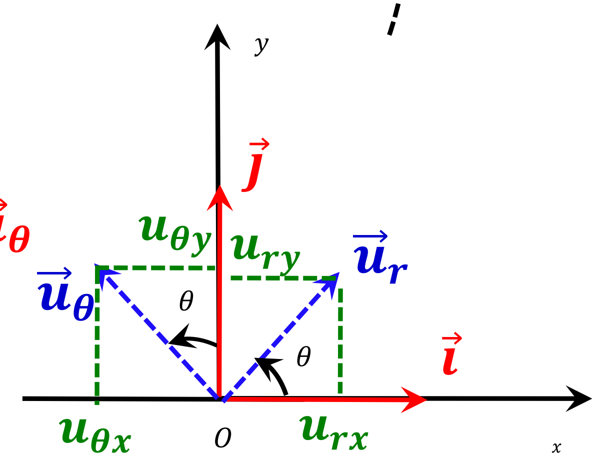


Calcul of $\frac{d\vec{u}_r}{dt}$:

$$\overrightarrow{u_r} = u_{rx}\vec{i} + u_{ry}\vec{j} = u_r\cos\theta\,\vec{i} + u_r\sin\theta\,\vec{j} = \cos\theta\,\vec{i} + \sin\theta\,\vec{j}$$

$$\frac{d\overrightarrow{u_r}}{dt} = -\sin\theta\frac{d\theta}{dt}\vec{i} + \cos\theta\frac{d\theta}{dt}\vec{j} = \frac{d\theta}{dt}(-\sin\theta\,\vec{i} + \cos\theta\,\vec{j})$$

$$\overrightarrow{u_\theta} = -u_{\theta x}\vec{i} + u_{\theta y}\vec{j} = -\sin\theta\,\vec{i} + \cos\theta\,\vec{j} \Rightarrow \frac{d\vec{u}_r}{dt} = \frac{d\theta(t)}{dt}\vec{u}_\theta$$



$$\boxed{\Rightarrow \vec{V}(t) = \frac{dr(t)}{dt}\vec{u}_r + r(t)\frac{d\theta(t)}{dt}\vec{u}_\theta}$$

$$\begin{cases} V_r(t) = \dfrac{dr(t)}{dt} & \textit{Radial component} \\ V_\theta(t) = r(t)\dfrac{d\theta(t)}{dt} & \textit{Transverse component} \end{cases}$$

Avec $|\vec{V}| = \sqrt{{V_r}^2 + {V_\theta}^2}$

En coordonnées cylindrique :

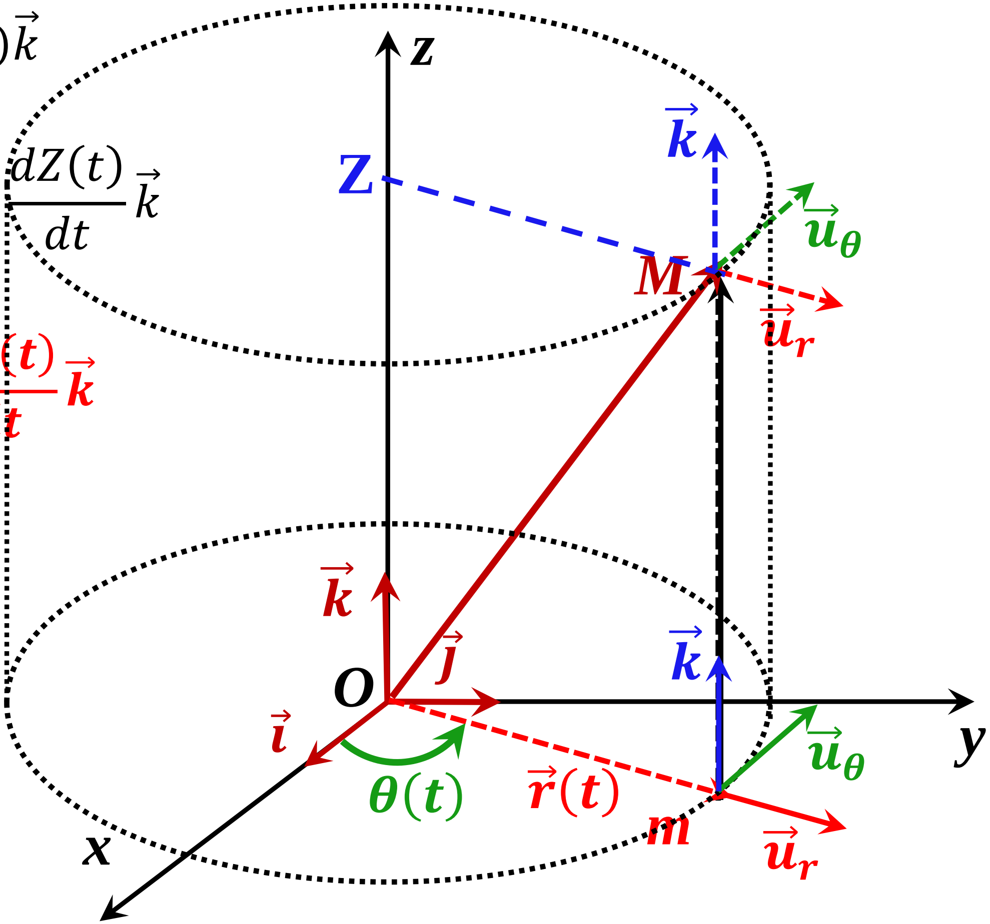
$$\overrightarrow{OM}(t) = \overrightarrow{Om}(t) + \overrightarrow{mM}(t) = r(t)\overrightarrow{u_r} + Z(t)\vec{k}$$

$$\vec{V}(t) = \frac{d\overrightarrow{OM}(t)}{dt} = \frac{dr(t)}{dt}\vec{u}_r + \vec{r}(t)\frac{d\vec{u}_r}{dt} + \frac{dZ(t)}{dt}\vec{k}$$

$$\Rightarrow \vec{V}(t) = \frac{dr(t)}{dt}\vec{u}_r + r(t)\frac{d\theta(t)}{dt}\vec{u}_\theta + \frac{dZ(t)}{dt}\vec{k}$$

$$\begin{cases} V_r(t) = \dfrac{dr(t)}{dt} \\ V_\theta(t) = r(t)\dfrac{d\theta(t)}{dt} \\ V_z(t) = \dfrac{dZ(t)}{dt} \end{cases}$$

$$\|\vec{V}\| = \sqrt{{V_r}^2 + {V_\theta}^2 + {V_Z}^2}$$



$$\begin{cases} r(t)\text{: Rayonpolaire}(r: 0 \to \infty) \\ \theta(t)\text{: Anglepolaire}(\theta: 0 \to 2\pi) \\ Z(t)\text{: Cote}(z: -\infty \to +\infty) \end{cases}$$

Intrinsic Coordinates of Velocity : (Tangential and Normal compents)

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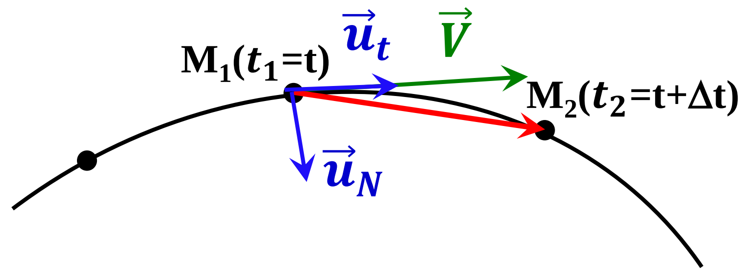


$$\vec{V}_{inst}(t) = \vec{V}(t) = \lim_{t_2 \to t_1} \frac{\overrightarrow{M_1M_2}(t)}{t_2 - t_1} = \frac{d\overrightarrow{OM}(t)}{dt}$$

Let's call $\widehat{M_1M_2}(t)$ length of arc having from M1 to M2

We have:

$$\vec{V}(t) = \lim_{t_2 \to t_1} \frac{\overrightarrow{M_1M_2}(t)}{\widehat{M_1M_2}(t)} \frac{\widehat{M_1M_2}(t)}{t_2 - t_1} = \lim_{t_2 \to t_1} \frac{\overrightarrow{M_1M_2}(t)}{\widehat{M_1M_2}(t)} \lim_{t_2 \to t_1} \frac{\widehat{M_1M_2}(t)}{t_2 - t_1}$$

When: $t_1 \to t_2 \Rightarrow \widehat{M_1M_2}(t) \to \left\|\overrightarrow{M_1M_2}(t)\right\| \quad \Rightarrow \frac{\overrightarrow{M_1M_2}(t)}{\widehat{M_1M_2}(t)} = \vec{u}_t$

$$\Longrightarrow \vec{V}(t) = \frac{d\widehat{M_1M_2}(t)}{dt}\vec{u}_t = \frac{dS(t)}{dt}\vec{u}_t$$

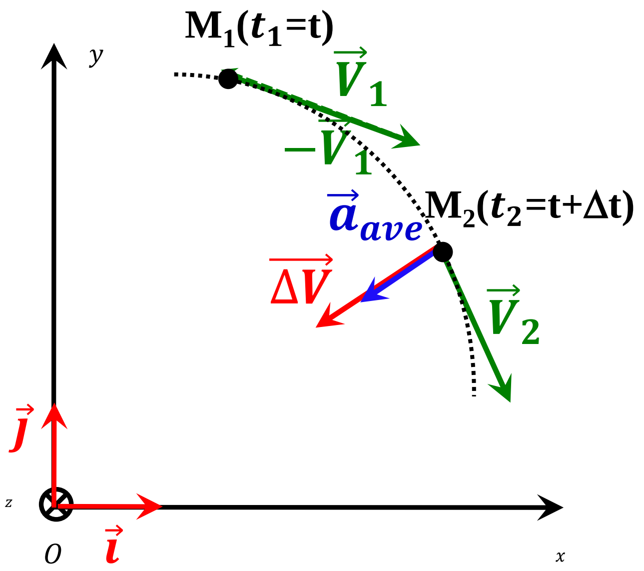
The **intrinsic coordinate system** for each point of the trajectory is defined as a system of reference formed by ***two axes***:

- ***Tangent axis***: its direction is *tangent* to the trajectory and is positive in the same direction than the velocity at that point. It is defined by the unit vector $\vec{u}_t$
- ***Normal axis***: it is *perpendicular* to the trajectory and is positive toward the center of curvature of the trajectory. It is defined by the unit vector $\vec{u}_N$

This reference system is used to "observe" the changes in the magnitude and *direction* of the *velocity vector*.

➢ Average Acceleration:

$$\vec{a}_{ave} = \frac{\vec{V}_2(t) - \vec{V}_1(t)}{t_2 - t_1} = \frac{\overrightarrow{\Delta V}(t)}{\Delta t}$$



In Cartesian coordinates:

$$\vec{V}_1 \begin{pmatrix} V_{x1} \\ V_{y1} \\ V_{z1} \end{pmatrix}, \vec{V}_2 \begin{pmatrix} V_{x2} \\ V_{y2} \\ V_{z3} \end{pmatrix}$$

$$\vec{a}_{ave} = \frac{V_{x_2} - V_{x_1}}{t_2 - t_1}\vec{i} + \frac{V_{y_2} - V_{y_1}}{t_2 - t_1}\vec{j} + \frac{V_{z_2} - V_{z_1}}{t_2 - t_1}\vec{k}$$

$$\vec{a}_{ave} = \frac{\Delta V_x}{\Delta t}\vec{i} + \frac{\Delta V_y}{\Delta t}\vec{j} + \frac{\Delta V_z}{\Delta t}\vec{k}$$

$$\begin{cases} \Delta V_x = V_{x_2} - V_{x_1} \\ \Delta V_y = V_{y_2} - V_{y_1} \\ \Delta V_z = V_{z_2} - V_{z_1} \end{cases}$$

Instantaneous acceleration

$$\vec{a}_{inst} = \vec{a} = \lim_{\Delta t \to 0} \vec{a}_{ave} = \lim_{\Delta t \to 0} \frac{\overrightarrow{\Delta V}(t)}{\Delta t} = \lim_{t_2 \to t_1} \frac{\vec{V}_2(t) - \vec{V}_1(t)}{t_2 - t_1} = \frac{\overrightarrow{dV}(t)}{dt}$$

In Cartesian coordinates:

$$\vec{a}_{inst} = \frac{dV_x}{dt}\vec{i} + \frac{dV_y}{dt}\vec{j} + \frac{dV_z}{dt}\vec{k}$$

$$\begin{cases} V_x = \dfrac{dx}{dt} \\ V_y = \dfrac{dy}{dt} \\ V_z = \dfrac{dz}{dt} \end{cases} \Longrightarrow \vec{a} = \frac{d^2x}{dt^2}\vec{i} + \frac{d^2y}{dt^2}\vec{j} + \frac{d^2z}{dt^2}\vec{k}$$

$$\begin{cases} a_x = \dfrac{dV_x}{dt} = \dfrac{d^2x}{dt^2} \\ a_y = \dfrac{dV_y}{dt} = \dfrac{d^2y}{dt^2} \\ a_z = \dfrac{dV_z}{dt} = \dfrac{d^2z}{dt^2} \end{cases}$$

$$\|\vec{a}\| = a = \sqrt{a_x{}^2 + a_y{}^2 + a_z{}^2}$$

In Polar coordinates:

$$\vec{V}(t) = \frac{dr(t)}{dt}\vec{u}_r + r(t)\frac{d\theta(t)}{dt}\vec{u}_\theta \quad \Rightarrow \vec{a}(t) = \frac{\overrightarrow{dV}}{dt} = \frac{d}{dt}\left(\frac{\boldsymbol{dr(t)}}{\boldsymbol{dt}}\vec{\boldsymbol{u}}_{\boldsymbol{r}}\right) + \frac{d}{dt}\left(\boldsymbol{r(t)}\frac{\boldsymbol{d\theta(t)}}{\boldsymbol{dt}}\vec{\boldsymbol{u}}_{\boldsymbol{\theta}}\right)$$

$$= \frac{d^2r(t)}{dt^2}\vec{u}_r + \frac{dr(t)}{dt}\underbrace{\frac{d\vec{u}_r}{dt}}_{\frac{d\theta(t)}{dt}\vec{u}_\theta} + \frac{dr(t)}{dt}\frac{d\theta(t)}{dt}\vec{u}_\theta + r(t)\frac{d^2\theta(t)}{dt^2}\vec{u}_\theta + r(t)\frac{d\theta(t)}{dt}\underbrace{\frac{d\vec{u}_\theta}{dt}}_{-\frac{d\theta(t)}{dt}\vec{u}_r}$$

$$\Rightarrow \vec{a} = \frac{d^2r(t)}{dt^2}\vec{u}_r + \frac{dr(t)}{dt}\frac{d\theta(t)}{dt}\vec{u}_\theta + \frac{dr(t)}{dt}\frac{d\theta(t)}{dt}\vec{u}_\theta + r(t)\frac{d^2\theta(t)}{dt^2}\vec{u}_\theta - r(t)\frac{d\theta(t)}{dt}\frac{d\theta(t)}{dt}\vec{u}_r$$

$$\Rightarrow \vec{\boldsymbol{a}} = \underbrace{\left(\frac{\boldsymbol{d^2r(t)}}{\boldsymbol{dt^2}} - \boldsymbol{r(t)}\left(\frac{\boldsymbol{d\theta(t)}}{\boldsymbol{dt}}\right)^{\boldsymbol{2}}\right)}_{a_r}\vec{\boldsymbol{u}}_{\boldsymbol{r}} + \underbrace{\left(\boldsymbol{2}\frac{\boldsymbol{dr(t)}}{\boldsymbol{dt}}\frac{\boldsymbol{d\theta(t)}}{\boldsymbol{dt}} + \boldsymbol{r(t)}\frac{\boldsymbol{d^2\theta(t)}}{\boldsymbol{dt^2}}\right)}_{a_\theta}\vec{\boldsymbol{u}}_{\boldsymbol{\theta}}$$

In Cylindrical coordinates:

$$\vec{V}(t) = \frac{dr(t)}{dt}\vec{u}_r + r(t)\frac{d\theta(t)}{dt}\vec{u}_\theta + \frac{dZ}{dt}\vec{k}$$

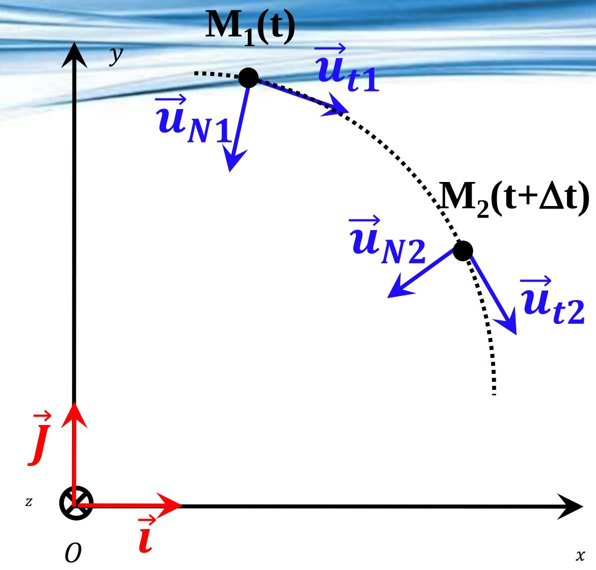
$$\Rightarrow \vec{a}(t) = \frac{d}{dt}\left(\boldsymbol{\frac{dr(t)}{dt}\vec{u}_r}\right) + \frac{d}{dt}\left(\boldsymbol{r(t)\frac{d\theta(t)}{dt}\vec{u}_\theta}\right) + \frac{d^2Z}{dt^2}\vec{k}$$

Using the same method, we find:

$$\Rightarrow \vec{a} = \underbrace{\left(\frac{d^2r(t)}{dt^2} - r(t)\left(\frac{d\theta(t)}{dt}\right)^2\right)}_{a_r}\vec{u}_r + \left(\underbrace{2\frac{dr(t)}{dt}\frac{d\theta(t)}{dt} + r(t)\frac{d^2\theta(t)}{dt^2}}_{a_\theta}\right)\vec{u}_\theta + \underbrace{\frac{d^2Z}{dt^2}}_{a_z}\vec{k}$$

$$\|\vec{a}\| = a = \sqrt{{a_r}^2 + {a_\theta}^2 + {a_z}^2}$$

Intrinsic compents of acceleration:



We'v : $\vec{V}(t) = \frac{dS(t)}{dt}\vec{u}_t$

$$\vec{a} = \frac{d\vec{V}}{dt} = \frac{d}{dt}\left(\frac{dS(t)}{dt}\vec{u}_t\right) = \frac{d^2S(t)}{dt^2}\vec{u}_t + \frac{dS(t)}{dt}\frac{d\vec{u}_t}{dt}$$

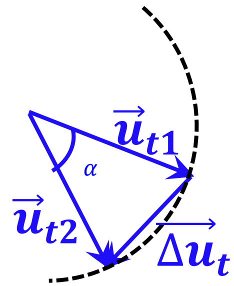
Calculation of $\frac{d\vec{u}_t}{dt}$ *:*

❑ Property :

The derivative of a unit vector is a vector orthogonal to that vector

$$\|\vec{u}_t\| = 1 \Rightarrow \frac{d\vec{u}_t}{dt} \perp \vec{u}_t \quad \Rightarrow \frac{d\vec{u}_t}{dt} = \frac{du_t}{dt}\vec{u}_N$$

$\|\vec{u}_{t1}\| = \|\vec{u}_{t2}\|$



On the other hand, we have:

$$\begin{cases} \overrightarrow{\Delta u_t} = \vec{u}_{t2} - \vec{u}_{t1} \\ \|\overrightarrow{\Delta u_t}\| = \|\vec{u}_{t1}\| sin\alpha \end{cases} \quad t_1 \to t_2 \Rightarrow \begin{cases} \overrightarrow{\Delta u_t} \to d\vec{u}_t \\ \alpha \to d\alpha \; (sind\alpha \approx d\alpha) \end{cases} \quad \Rightarrow \|d\vec{u}_t\| = \|\vec{u}_t\| d\alpha = d\alpha$$

$$\Rightarrow \frac{d\vec{u}_t}{dt} = \frac{du_t}{dt}\vec{u}_N = \frac{d\alpha}{dt}\vec{u}_N$$

$$\vec{a} = \frac{d^2S(t)}{dt^2}\vec{u}_t + \frac{dS(t)}{dt}\frac{d\alpha}{dt}\vec{u}_N = \frac{d^2S}{dt^2}\vec{u}_t + \frac{dS}{dt}\frac{d\alpha}{\boldsymbol{dS}}\frac{\boldsymbol{dS}}{dt}\vec{u}_N$$

$\frac{dS}{d\alpha} = \rho$: Trajectory Radius

$$\vec{a} = \underbrace{\frac{d^2S(t)}{dt^2}}_{\vec{a}_t}\vec{u}_t + \underbrace{\frac{1}{\rho}\left(\frac{dS(t)}{dt}\right)^2}_{\vec{a}_N}\vec{u}_N = \frac{dV(t)}{dt}\vec{u}_t + \frac{1}{\rho}V(t)^2\vec{u}_N$$

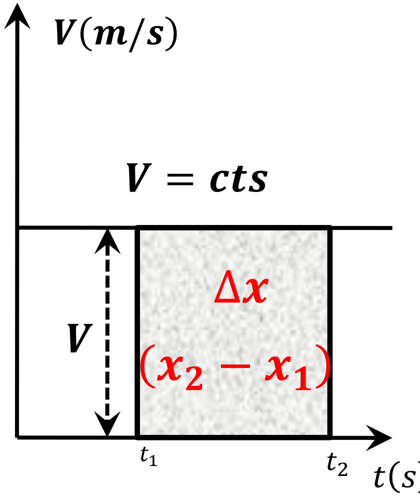
$$\begin{cases} \frac{d^2S}{dt^2} = \frac{dV}{dt}: \textit{Tangential component of } \vec{a} \textit{ related to the change in modulus of } \vec{V} \\ \frac{1}{\rho}\frac{dS}{dt} = \frac{1}{\rho}V: \textit{Normal component of } \vec{a} \textit{ related to the change in direction of } \vec{V} \end{cases}$$

$$\vec{a} = a_t\vec{u}_t + a_N\vec{u}_N \qquad \|\vec{a}\| = a = \sqrt{a_t^2 + a_N^2}$$

II.3. Transition from speed to distance travelled – Integral calculation :

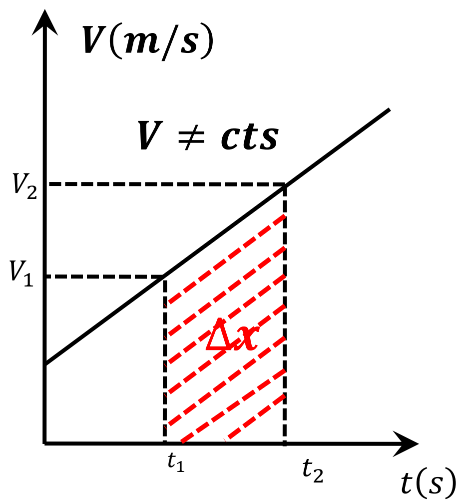
❖*Let be a mobile "**M**" moving with a constant velocity in rectilinear motion*

$$\Rightarrow V_{moy} = V_{inst} = V = \frac{\Delta x}{\Delta t} = \frac{x_2 - x_1}{t_2 - t_1}$$



Knowing V and x_1 at $t = t_1$ $\Rightarrow x_2 = x_1 + V(t_2 - t_1)$

The distance Δx traveled between t_1 and t_2 is measured by the area under the curve $V(t)$: $\Delta x = V(t_2 - t_1)$

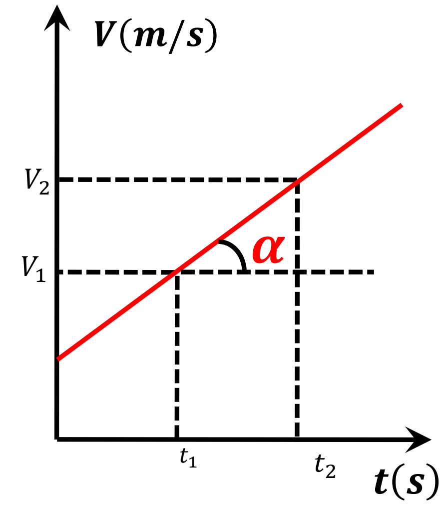


❖ **When the velocity is not constant Δx is always equal to the area under the curve $V(t)$** $\left(\Delta x = (V_2 - V_1)(t_2 - t_1)\right)$

II.4- Transition from acceleration to velocity:

If the motion is defined by the given acceleration, the velocity is equal to the integral of the acceleration (acceleration is the derivative of velocity).

$$a = \frac{dV}{dt} \Rightarrow \int_{V_1}^{V_2} dV = \int_{t_1}^{t_2} a dt$$



Geometrically:

- **The acceleration will be the curve tangent of V(t)**

$$a = tg\alpha = \frac{V_2 - V_1}{t_2 - t_1}$$

Exemple :

An object moves in an oriented straight line with a velocity that obeys the law:

$$a = 4 - t^2 (m/s^2)$$

- Find, as a function of time, the expressions for velocity and position.

We give: $t = 3s \Rightarrow V = 2m/s, x = 9m.$

- Represent the velocity and acceleration vectors at **t = 1s.**

Solution :

1. $V = \int a dt \quad = \int (4 - t^2) dt \quad = 4t - \frac{t^3}{3} \quad +C$

$t = 3s \Rightarrow V = 2m/s \quad \Rightarrow 2 = 4.3 - \frac{3^3}{3} + C \quad \Rightarrow C = -1 \quad \boxed{\Rightarrow V = 4t - \frac{t^3}{3} - 1}$

$x = \int V dt \quad = \int \left(4t - \frac{t^3}{3} - 1\right) dt \quad = 2t^2 - \frac{1}{12} t^4 - t + C'$

$t = 3s \Rightarrow x = 9m \Rightarrow C' = 3/4 \quad \boxed{\Rightarrow x = -\frac{1}{12} t^4 + 2t^2 - t + 3/4}$

2.

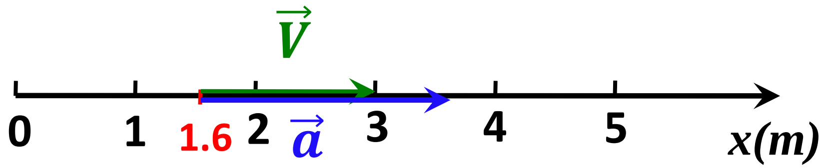
$$\begin{cases} x = -\frac{1}{12}t^4 + 2t^2 - t + 3/4 \\ V = -\frac{t^3}{3} + 4t - 1 \\ a = 4 - t^2 \end{cases}$$

$$\Rightarrow t = 1s: \begin{cases} x = -\frac{1}{12} + 2 - 1 + 3/4 = 1.6m \\ V = -\frac{1}{3} + 4 - 1 = 2.6m/s \\ a = 4 - 1 = 3m/s^2 \end{cases}$$

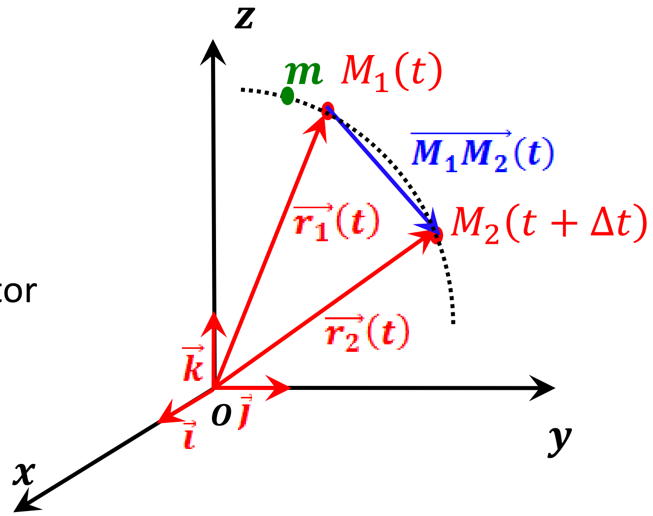
Echelle : $x: 1cm \rightarrow 1m$

$V: 1cm \rightarrow 2m/s$

$a: 1cm \rightarrow 1.5m/s^2$



Summary



- $\widehat{M_1M_2}(t) = S(t)$:Curvilinear Coordinate
- $\overrightarrow{OM_1}$ et $\overrightarrow{OM_2}$: is a Position Vectors of « M » /« O »
- $\left|\overrightarrow{M_1M_2}(t)\right| = \left|\Delta\overrightarrow{OM}(t)\right| = \left|\overrightarrow{OM_2}(t) - \overrightarrow{OM_1}(t)\right|$: Displacement vector

Velocity (m/s)

Average Velocity $(\vec{V}_{ave}(t))$	Instantaneous velocity $\vec{V}_{inst} = \vec{V}$
$\dfrac{\Delta\overrightarrow{OM}(t)}{\Delta t} = \dfrac{\overrightarrow{M_1M_2}(t)}{\Delta t} = \dfrac{\Delta x}{\Delta t}\vec{i} + \dfrac{\Delta y}{\Delta t}\vec{j} + \dfrac{\Delta z}{\Delta t}\vec{k}$	$\lim\limits_{\Delta t\to 0} \vec{V}_{ave} = \lim\limits_{\Delta t\to 0} \dfrac{\Delta\overrightarrow{OM}(t)}{\Delta t} = \dfrac{d\overrightarrow{OM}(t)}{dt}$

Acceleration

Average Acceleration$(\vec{a}_{ave}(t))$	Instantaneous Acceleration $\vec{a}_{inst} = \vec{a}$
$\dfrac{\Delta\vec{V}(t)}{\Delta t} = \dfrac{\Delta V_x}{\Delta t}\vec{i} + \dfrac{\Delta V_y}{\Delta t}\vec{j} + \dfrac{\Delta V_z}{\Delta t}\vec{k}$	$\lim\limits_{\Delta t\to 0} \vec{a}_{ave} = \lim\limits_{\Delta t\to 0} \dfrac{\Delta\vec{V}(t)}{\Delta t} = \dfrac{d\vec{V}(t)}{dt}$

	Cartesian Coordinates $(\vec{i},\vec{j},\vec{k})$	Polar Coordinates $(\vec{u}_r,\ \vec{u}_\theta)$	Cylindric Coordinates $(\vec{u}_r,\ \vec{u}_\theta,\ \vec{k})$
$\overrightarrow{OM}(t)$	$x(t)\vec{i}+y(t)\vec{j}+z(t)\vec{k}$	$r(t)\vec{u}_r$	$r(t)\vec{u}_r+z(t)\vec{k}$
$\vec{V}(t)$	$\frac{dx}{dt}\vec{i}+\frac{dy}{dt}\vec{j}+\frac{dz}{dt}\vec{k}$	$\frac{dr(t)}{dt}\vec{u}_r+r(t)\frac{d\theta(t)}{dt}\vec{u}_\theta$	$\frac{dr(t)}{dt}\vec{u}_r+r(t)\frac{d\theta(t)}{dt}\vec{u}_\theta+\frac{dz(t)}{dt}\vec{k}$
$\vec{a}$	$\frac{d^2x}{dt^2}\vec{i}+\frac{d^2y}{dt^2}\vec{j}+\frac{d^2z}{dt^2}\vec{k}$	$\left(\frac{d^2r(t)}{dt^2}-r(t)\left(\frac{d\theta(t)}{dt}\right)^2\right)\vec{u}_r+\left(2\frac{dr(t)}{dt}\frac{d\theta(t)}{dt}+r(t)\frac{d^2\theta(t)}{dt^2}\right)\vec{u}_\theta$	$\left(\frac{d^2r(t)}{dt^2}-r(t)\left(\frac{d\theta(t)}{dt}\right)^2\right)\vec{u}_r+\left(2\frac{dr(t)}{dt}\frac{d\theta(t)}{dt}+r(t)\frac{d^2\theta(t)}{dt^2}\right)\vec{u}_\theta+\frac{d^2Z}{dt^2}\vec{k}$

Intrinsec coordinates $(\vec{u}_t,\ \vec{u}_N)$	
Velocity	$\vec{V}(t)=\frac{dS(t)}{dt}\vec{u}_t=V(t)\vec{u}_t$
Acceleration	$\vec{a}(t)=\frac{dV(t)}{dt}\vec{u}_t+\frac{1}{\rho}V(t)^2\vec{u}_N=a_t\vec{u}_t+a_N\vec{u}_N$

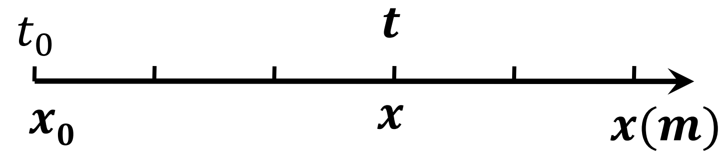
III. Some specific movements

III.1. Rectilinear motion:

In this type of motion, the trajectories are straight lines and the position of the mobile is described by a single coordinate $x(t)$ equivalent to the path traveled $S(t)$.

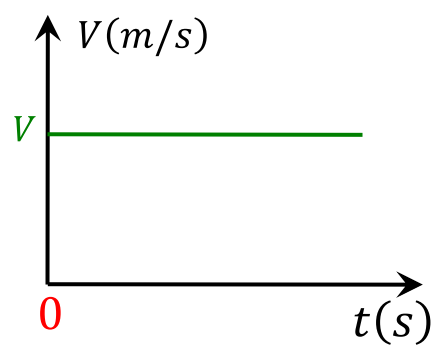
III.1.1. Uniform Rectilinear Motion (URM): الحركة المستقيمة المنتظمة

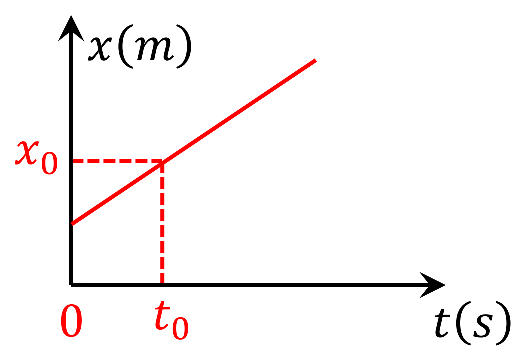
Characterized by *V(t) = Cts = V*



$$V = \frac{dx}{dt} \Rightarrow dx = Vdt \quad \Rightarrow \int_{x_0}^{x} dx = \int_{t_0}^{t} Vdt \quad \Rightarrow x - x_0 = V(t - t_0)$$

$\Rightarrow$ ***Equation of Motion :*** $\boxed{x(t) = V(t - t_0) + x_0}$





III.1.2. Uniformly Varied Rectilinear Motion(UVRM):

الحركة المستقيمة المتغيرة بإنتظام

Characterize by $\boldsymbol{a(t) = Cts = a}$

$$t = t_0 : \begin{cases} x = x_0 \\ V = V_0 \end{cases}$$

$$a = \frac{dV}{dt} \Rightarrow dV = a dt \quad \Rightarrow \int_{V_0}^{V} dV = a \int_{t_0}^{t} dt \quad \Rightarrow V - V_0 = a(t - t_0)$$

$$\boxed{\Rightarrow V(t) = a(t - t_0) + V_0}$$

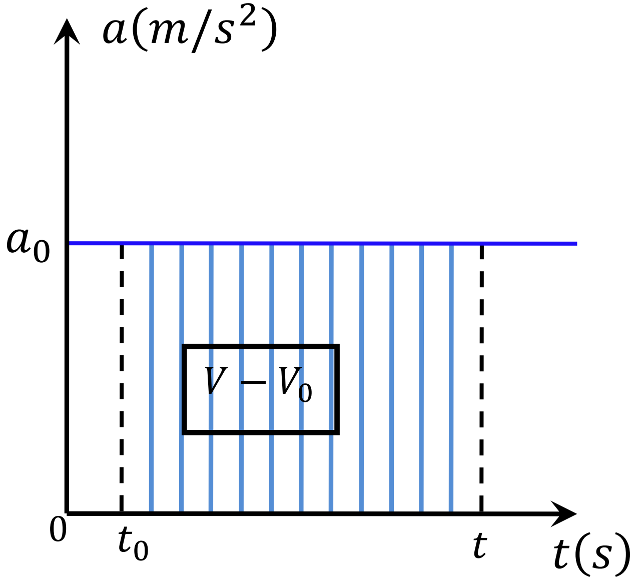
- **Equation of Motion**

$$V(t) = \frac{dx}{dt} \Longrightarrow \int_{x_0}^{x} dx = \int_{t_0}^{t} V dt \quad = \int_{t_0}^{t} (a(t - t_0) + V_0) dt$$

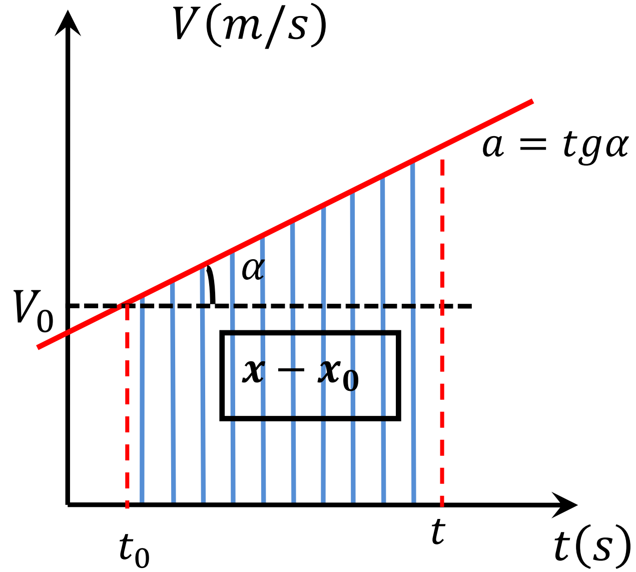
$$\Rightarrow x - x_0 = \frac{1}{2} a(t - t_0)^2 + V_0(t - t_0)$$

$$\boxed{\Rightarrow x = \frac{1}{2} a(t - t_0)^2 + V_0(t - t_0) + x_0}$$

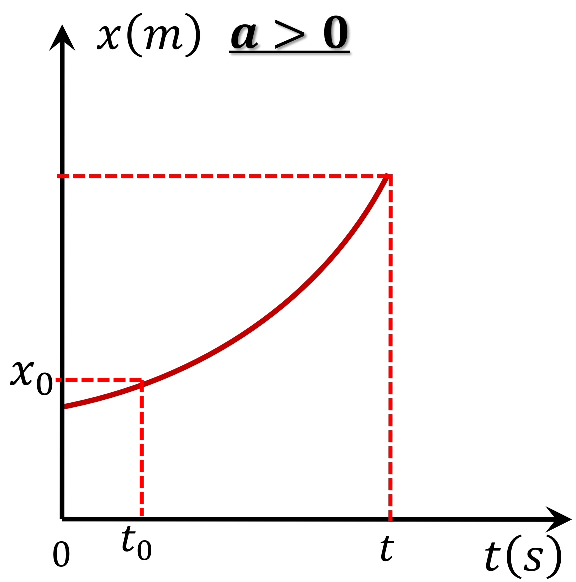
(Equation of Motion)



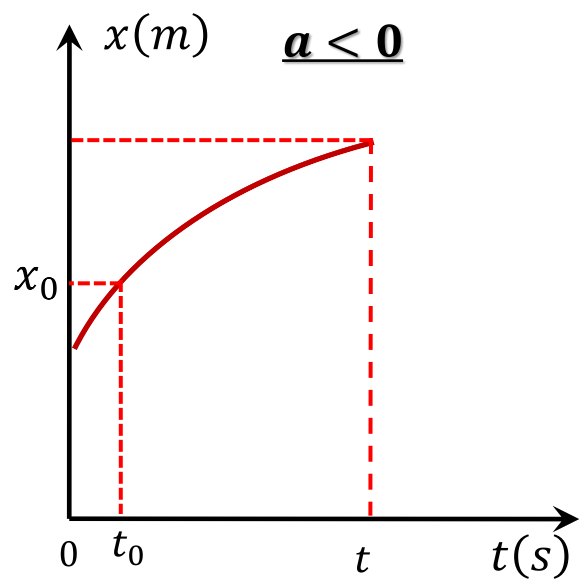
$a(m/s^2)$
a_0
$V - V_0$
0
t_0
t
$t(s)$



$V(m/s)$
$a = tg\alpha$
α
V_0
$x - x_0$
t_0
t
$t(s)$



$x(m)$
$a > 0$
x_0
0
t_0
t
$t(s)$



$x(m)$
$a < 0$
x_0
0
t_0
t
$t(s)$

Remark:

The acceleration or deceleration of a uniformly varying motion is defined by the sign of the dot product $\vec{a}.\vec{V}$:

$\vec{a}.\vec{V}$ **>0** : **Two possible cases:**

$\vec{a} > 0 \; et \; \vec{V} > 0$: M. Accelerated ***Uniformly*** in the positive direction of motion

$\vec{a} < 0 \; et \; \vec{V} < 0$: M. Accelerated ***Uniformly*** in the negative direction of motion

$\vec{a}.\vec{V}$**< 0** : **Two possible cases :**

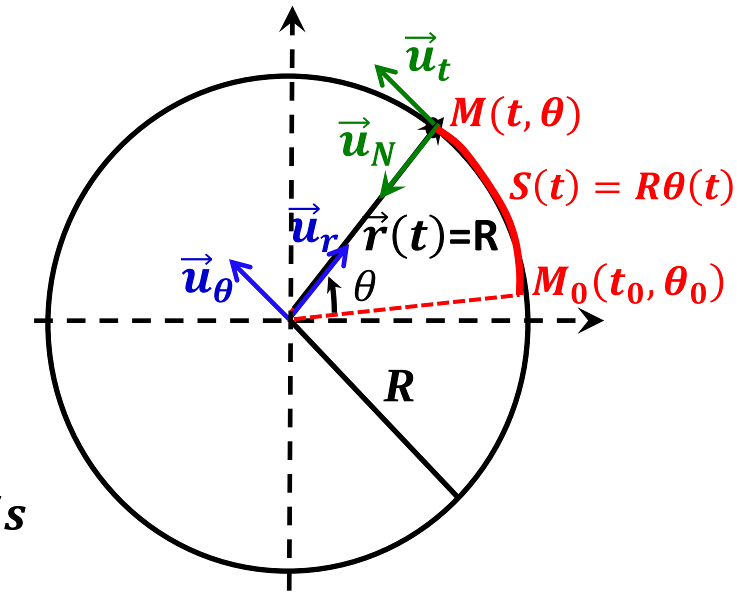
$\vec{a} < 0 \; et \; \vec{V} > 0$: M. decelerated ***Uniformly*** in the positive direction of motion

$\vec{a} > 0 \; et \; \vec{V} < 0$: M. decelerated ***Uniformly*** in the negative direction of motion

III.2. Circular Motion:

This type of motion is characterized by a circular trajectory with a constant radius :

$$\boldsymbol{r(t) = cte = R}$$



In polar coordinates: $\vec{r}(t) = R\vec{u}_r$

$$\vec{V}(t) = \frac{dr(t)}{dt}\vec{u}_r + r(t)\frac{d\theta(t)}{dt}\vec{u}_\theta = \cancelto{0}{\frac{dR}{dt}}\vec{u}_r + R\frac{d\theta(t)}{dt}\vec{u}_\theta$$

$$\Rightarrow \boldsymbol{\vec{V}(t) = R\frac{d\theta(t)}{dt}\vec{u}_\theta}$$

$$\boldsymbol{\frac{d\theta(t)}{dt} = \omega(t)}$$: ***Called Angular Velocity***, $\boldsymbol{[\omega] = rad/s}$

$$\boxed{\Rightarrow \boldsymbol{\vec{V}(t) = R\omega(t)\vec{u}_\theta}}$$

En coordonnées intrinsèques : $\vec{V}(t) = \frac{dS(t)}{dt}\vec{u}_t$

$$S(t) = R\theta(t) \quad \Rightarrow \vec{V}(t) = R\frac{d\theta(t)}{dt}\vec{u}_t = R\omega(t)\vec{u}_t$$

$$\boxed{\Rightarrow \boldsymbol{\vec{V}(t) = R\omega(t)\vec{u}_\theta = R\omega(t)\vec{u}_t}}$$

Acceleration Expression:

- **In intrinsic coordinates:** $\vec{a} = a_t\vec{u}_t + a_N\vec{u}_N = \frac{dV(t)}{dt}\vec{u}_t + \frac{1}{\rho}V^2\vec{u}_N$

$$\begin{cases} \rho = R \\ V = R\omega \\ \omega = \frac{d\theta}{dt} \end{cases} \Rightarrow \vec{a}(t) = R\frac{d\omega(t)}{dt}\vec{u}_t + R\omega^2(t)\vec{u}_N = \boldsymbol{R\frac{d^2\theta(t)}{dt^2}\vec{u}_t + R\left(\frac{d\theta(t)}{dt}\right)^2\vec{u}_N}$$

- **In polar coordinates:**

$$\vec{a} = \left(\frac{d^2r(t)}{dt^2} - r(t)\left(\frac{d\theta(t)}{dt}\right)^2\right)\vec{u}_r + \left(2\frac{dr(t)}{dt}\frac{d\theta(t)}{dt} + r(t)\frac{d^2\theta(t)}{dt^2}\right)\vec{u}_\theta$$

$$r(t) = R \Rightarrow \vec{a} = \left(\overbrace{\frac{d^2R}{dt^2}}^{0} - R\left(\frac{d\theta(t)}{dt}\right)^2\right)\vec{u}_r + \left(2\overbrace{\frac{dR}{dt}}^{0}\frac{d\theta(t)}{dt} + R\frac{d^2\theta(t)}{dt^2}\right)\vec{u}_\theta$$

$$\boxed{\boldsymbol{\vec{a} = -R\left(\frac{d\theta(t)}{dt}\right)^2\vec{u}_r + R\frac{d^2\theta(t)}{dt^2}\vec{u}_\theta}} \quad \boxed{\boldsymbol{= -R\omega^2(t)\vec{u}_r + R(t)\frac{d\omega(t)}{dt}\vec{u}_\theta}}$$

$\boldsymbol{\frac{d\omega(t)}{dt}} = \alpha$: **Angular acceleration,** $[\alpha] = \boldsymbol{rad/s^2}$

III.2.1. Uniform Circular Motion(UCM): الحركة الدائرية المنتظمة

This type of motion is characterized by a constant angular velocity:

$$V(t) = Cst = R\omega(t) \Rightarrow \omega(t) = Cst$$

$$\vec{a} = \begin{cases} a_t = R\dfrac{d\omega(t)}{dt} = 0 \\ a_N = R\omega^2 \end{cases} \quad \textbf{Or} \quad \vec{a} = \begin{cases} a_r = -R\omega^2(t) \\ a_\theta = R\dfrac{d\omega(t)}{dt} = 0 \end{cases}$$

- ***Equation of motion:***

$$\omega = \frac{d\theta(t)}{dt} \Rightarrow d\theta(t) = \omega dt \Rightarrow \int_{\theta_0}^{\theta(t)} d\theta(t) = \int_{t_0}^{t} \omega\, dt \implies \theta(t) - \theta_0 = \omega\,(t - t_0)$$

So the equation of this motion is given by:

$$\boxed{\boldsymbol{\theta(t) = \omega(t - t_0) + \theta_0}}$$

III.2.2. Uniformly Variable Circular Motion (UVCM): الحركة الدائرية المتغيرة بانتظام

This type of motion is characterized by constant tangential acceleration: $a_t(t) = Cst$

$$a_t(t) = R\frac{d\omega(t)}{dt} = Cst \qquad \Rightarrow \frac{d\omega(t)}{dt} = \alpha = Cst \qquad \Rightarrow d\omega(t) = \alpha dt \qquad \left(\boldsymbol{t = t_0}\text{:} \begin{cases} \boldsymbol{\omega = \omega_0} \\ \boldsymbol{\theta = \theta_0} \end{cases}\right)$$

$$\Rightarrow \int_{\omega_0}^{\omega(t)} d\omega(t) = \int_{t_0}^{t} \alpha dt \qquad \boxed{\Rightarrow \boldsymbol{\omega(t) = \alpha(t - t_0) + \omega_0}}$$

On the other hand, we have:

$$\omega(t) = \frac{d\theta(t)}{dt} \quad \Rightarrow \int_{\theta_0}^{\theta(t)} d\theta(t) = \int_{t_0}^{t} \omega(t)dt \quad = \int_{t_0}^{t} (\alpha(t - t_0) + \omega_0)dt$$

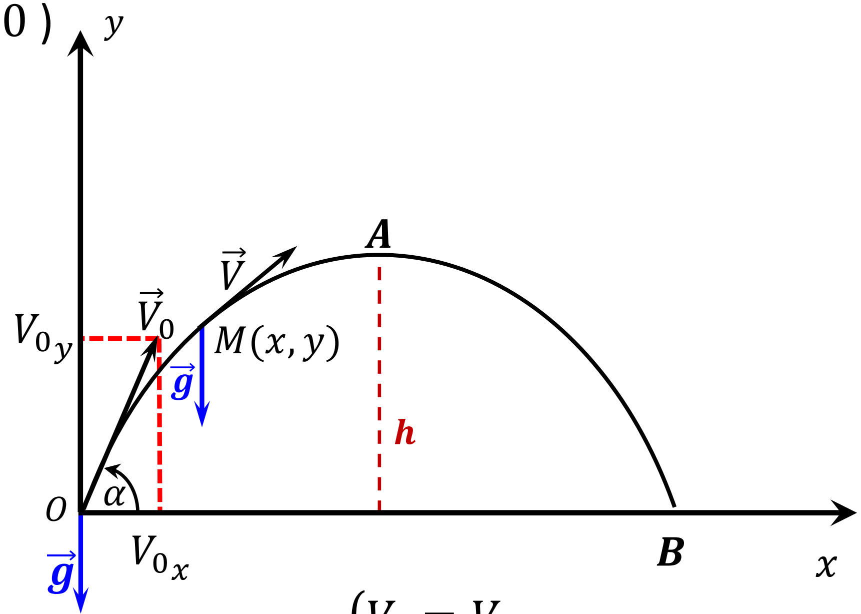
$$\Rightarrow \theta(t) - \theta_0 = \frac{1}{2}\alpha(t - t_0)^2 + \omega_0(t - t_0)$$

$$\boxed{\boldsymbol{\theta(t) = \frac{1}{2}\alpha(t - t_0)^2 + \omega_0(t - t_0) + \theta_0}}$$

(Equation of UVCM)

Application : Motion of a projectile

- $\vec{V}_0$: The initial velocity of the projectile ($t_0 = 0$)



- In this case: $\vec{a} = \vec{g} = -g\vec{\jmath}$

$$\vec{V}_0 = V_{0_x}\vec{\imath} + V_{0_y}\vec{\jmath} \quad \text{Where} \begin{cases} V_{0_x} = V_0 \cos\alpha \\ V_{0_x} = V_0 \sin\alpha \end{cases}$$

- On the other hand, at a given point $M(x, y)$,

  we have: $\vec{V} = \vec{a}(t - t_0) + \vec{V}_0$

- At $t_0 = 0$:

$$V_x\vec{\imath} + V_y\vec{\jmath} = -gt\vec{\jmath} + V_{0_x}\vec{\imath} + V_{0_y}\vec{\jmath} = V_{0_x}\vec{\imath} + \left(V_{0_y} - gt\right)\vec{\jmath} \quad \Longrightarrow \begin{cases} V_x = V_{0_x} \\ V_y = V_{0_y} - gt \end{cases}$$

- Also, we have:

$$\overrightarrow{OM} = \vec{r} = x\vec{\imath} + y\vec{\jmath} \quad = \frac{1}{2}\vec{a}(t - t_0)^2 + \vec{V}_0(t - t_0) + \vec{r}_0$$

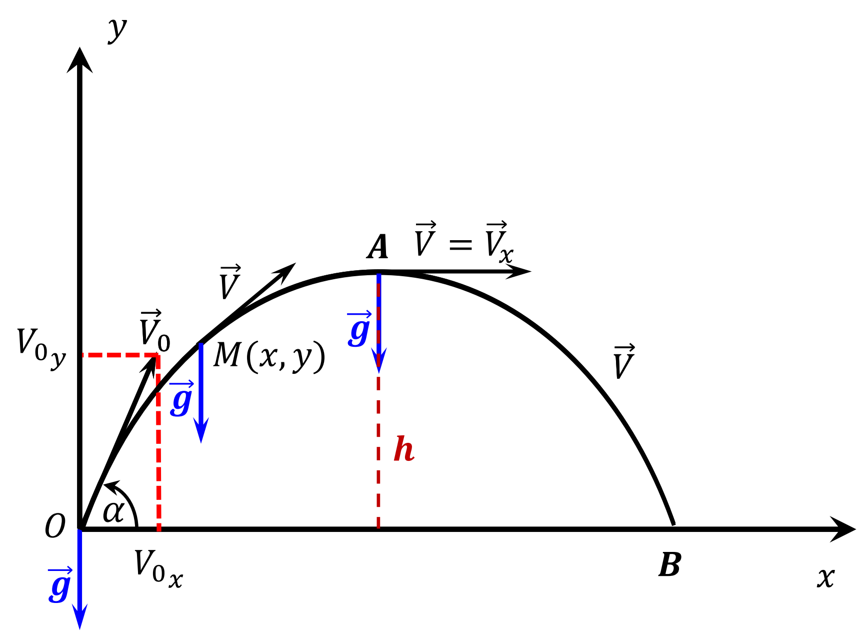
$$\text{At } t_0 = 0 : \begin{cases} r_0 = 0 \\ a = -g \end{cases} \Longrightarrow x\vec{\imath} + y\vec{\jmath} = -\frac{1}{2}\mathrm{g}t^2\vec{\jmath} + V_{0_x}t\vec{\imath} + V_{0_y}t\vec{\jmath} \Longrightarrow \begin{cases} x = V_{0_x}t \\ y = -\dfrac{1}{2}gt^2 + V_{0_y}t \end{cases}$$

❑ *The time requiered for the projectile to reach the highest Point **A** is obtained by setting*

$V_y = 0$. *in this point the velocity is horizontal*

<u>***Then:***</u> $V_y = 0 \Longrightarrow V_{0_y} - gt = 0 \quad \Longrightarrow t = \dfrac{V_{0_y}}{g}$

$$\boxed{\Longrightarrow \boldsymbol{t} = \frac{\boldsymbol{V_0 \sin\alpha}}{\boldsymbol{g}}}$$



❑ $\boldsymbol{h}$ is obtained by substituting this value of t in the equation of y: $y = -\dfrac{1}{2}gt^2 + V_{0_y}t$

$$\Longrightarrow h = -\frac{1}{2}g\frac{{V_0}^2 \sin^2\alpha}{g^2} + V_{0_y}\frac{V_0 \sin\alpha}{g}$$

$$= -\frac{1}{2}\frac{{V_0}^2 \sin^2\alpha}{g} + V_0 \sin\alpha\frac{V_0 \sin\alpha}{g}$$

$$\boxed{\Longrightarrow \boldsymbol{h} = \frac{1}{2}\frac{\boldsymbol{{V_0}^2 \sin^2\alpha}}{\boldsymbol{g}}}$$

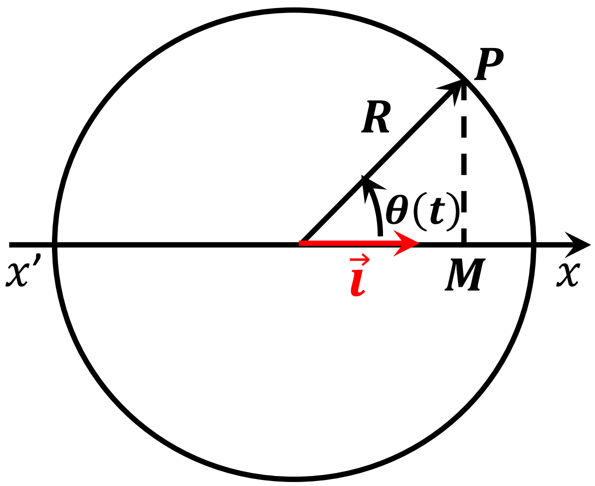
❑ *The time requiered for the projectile to return to ground level at point **B** can be obtainde by making* $y = 0$

$$\Longrightarrow -\frac{1}{2}gt^2 + V_{0_y}t = 0 \quad \Longrightarrow -\frac{1}{2}gt + V_0 \sin\alpha = 0 \qquad \boxed{\Longrightarrow \boldsymbol{t} = \frac{\boldsymbol{2V_0 \sin\alpha}}{\boldsymbol{g}}}$$

III.3. Harmonic Motion (Sinusoidal Rectilinear Motion): الحركة الجيبية

is consider as the projection, on a diameter, of an uniform circular motion of a point "P" of an angular velocity ω on a circle of radius R, With $\theta(t) = \omega\, t + \theta_0$

❑ Let "**M**" be the projection of "***P***" on $(x'x)$:



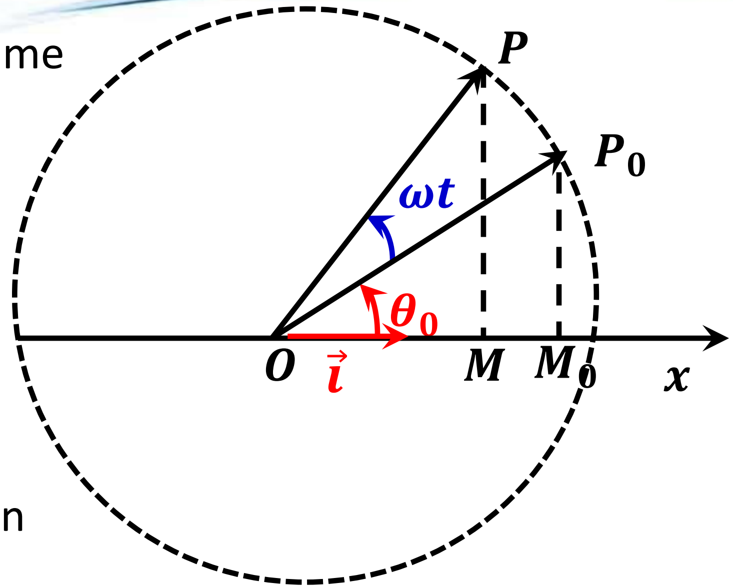
$$\overrightarrow{OM}(t) = x\vec{i} = R\cos\theta(t)\vec{i} = R\cos(\omega\, t + \theta_0)\vec{i}$$

$\omega\, t + \theta_0$: Motion Phase

θ_0 Initial phase or phase at the origin of time

$-R < x < +R$: ***is called amplitude***

- The motion of "P" reproduces itself identically each time that the angle ωt increases by $\boldsymbol{2\pi}$



$$T = \frac{2\pi}{\omega}$$: Presents the period of motion

- $\boldsymbol{\omega = 2\pi f}$: Pulsation or angular frequency **(rad/s)**

- $\boldsymbol{f = \frac{1}{T} = \frac{\omega}{2\pi}}$: is the frequency of motion ≡ Oscillation numbers per unit of time related to the period **(1/s, Hz)**

$$\vec{V} = V_x\vec{i} = \frac{dx}{dt}\vec{i} = \frac{d}{dx}\left(R\cos(\omega\, t + \theta_0)\right)\vec{i}$$

$$\boxed{\Longrightarrow \vec{V}(t) = -R\omega \sin(\omega\, t + \theta_0)\vec{i}}$$

$$\vec{a} = a_x\vec{i} = \frac{dV_x}{dt}\vec{i} = \frac{d}{dx}\left(-R\omega \sin(\omega\, t + \theta_0)\right)\vec{i} = -R\omega^2 \cos(\omega\, t + \theta_0)\vec{i}$$

$$\boxed{\Longrightarrow \vec{a}(t) = -\omega^2 x\vec{i}}$$

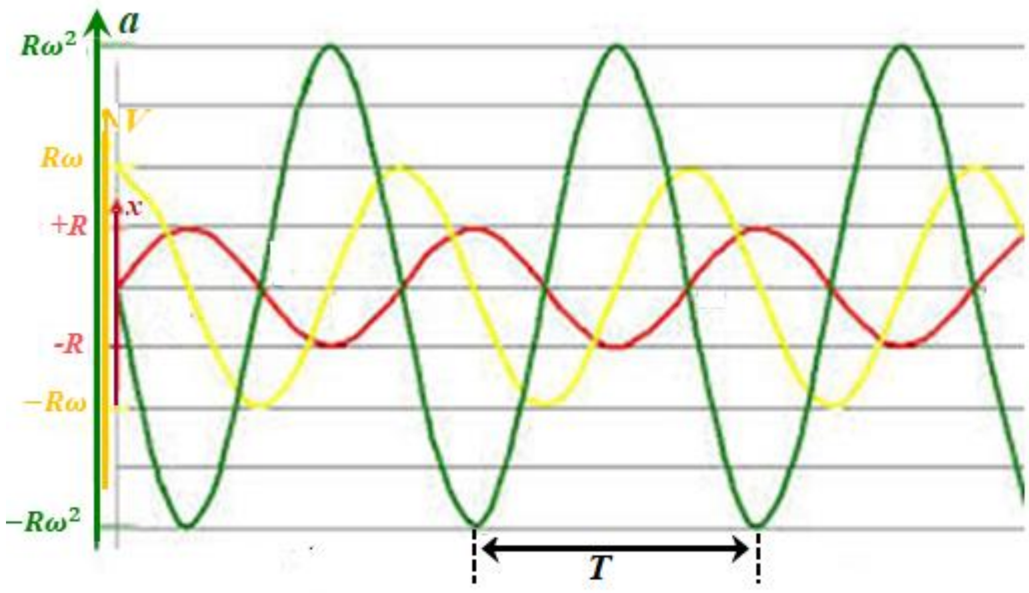
This indicates that the acceleration in harmonic motion is opposite to the position vector

$$\Rightarrow \vec{a} = -\omega^2\overrightarrow{OM}$$

$$\begin{cases} \overrightarrow{OM}(t) = Rcos(\omega t + \theta_0)\vec{i} \\ \vec{V} = -R\omega sin(\omega t + \theta_0)\vec{i} \\ \vec{a} = -R\omega^2 cos(\omega t + \theta_0)\vec{i} = -\omega^2\overrightarrow{OM} \end{cases},$$

- $\boldsymbol{\theta_0 = 0}$ **and** $\boldsymbol{t = 0}$**:** $\boldsymbol{cos(\omega t + \theta_0) = 0} \Rightarrow \begin{cases} \boldsymbol{OM = x = 0} \\ \boldsymbol{a = 0} \end{cases}$

$$\boldsymbol{sin(\omega t + \theta_0) = \pm 1} \Rightarrow \boldsymbol{V = \pm R\omega}$$



IV. Relative Motion

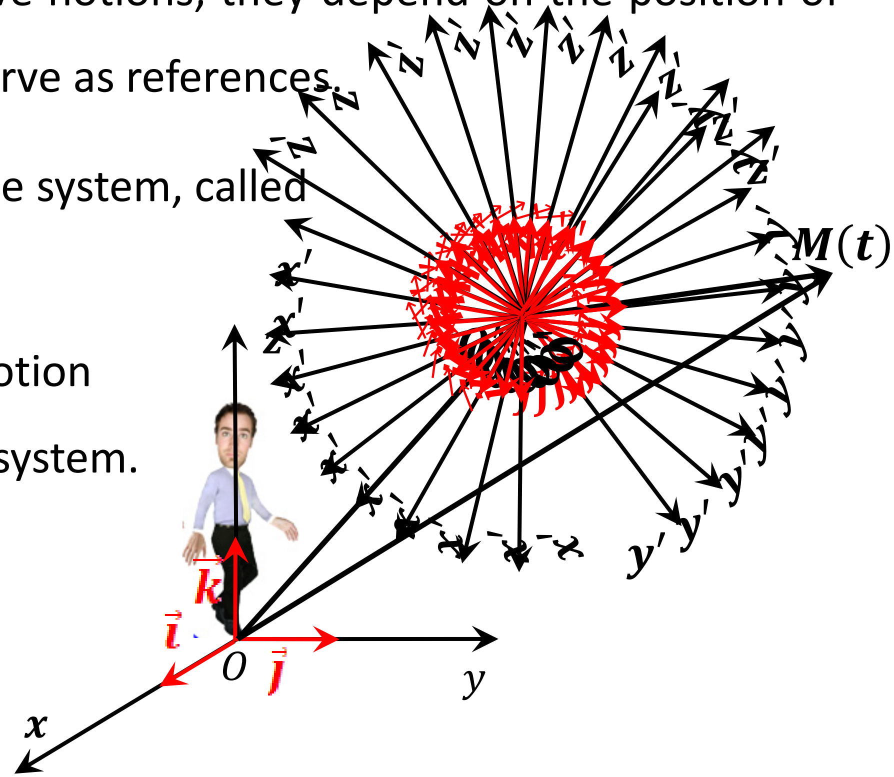
IV.1. Change of reference system:

In relative physics, rest, like motion, are relative notions, they depend on the position of the mobile in relation to other bodies which serve as references.

- Let $R(O, xyz)$ be a supposedly fixed coordinate system, called an absolute coordinate system.
- Let $R'(O', x'y'z')$ be a coordinate system in motion with respect to R, called a relative coordinate system.

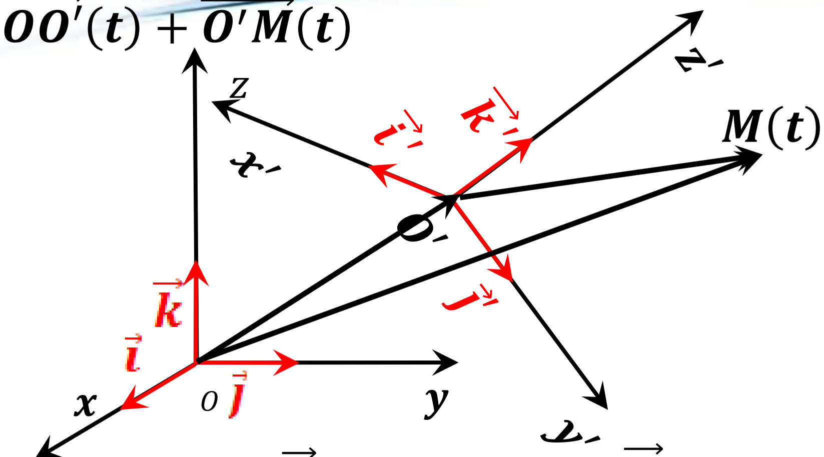
$$\overrightarrow{OM}(t)\Big/_{R} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$\overrightarrow{O'M}(t)\Big/_{R'} = x'\vec{i'} + y'\vec{j'} + z'\vec{k'}$$



To an observer bound to the R, the motion of R'(O'x'y'z') is known via the motion of O’/O, and the ways in which the axes Ox', Oy', and Oz' rotate around O'

- ***Relationship between positions:*** $\overrightarrow{\boldsymbol{OM}}(\boldsymbol{t}) = \overrightarrow{\boldsymbol{OO'}}(\boldsymbol{t}) + \overrightarrow{\boldsymbol{O'M}}(\boldsymbol{t})$



$$x\vec{i} + y\vec{j} + z\vec{k} = \overrightarrow{OO'} + x'\vec{i'} + y'\vec{j'} + z'\vec{k'}$$

Relation entre les vitesses:

$$\vec{V}(t) = \frac{d\overrightarrow{OM}(t)}{dt} = \frac{d\overrightarrow{OO'}(t)}{dt} + \frac{d\overrightarrow{O'M}(t)}{dt}$$

$$\Longrightarrow \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k} = \frac{d\overrightarrow{OO'}}{dt} + \frac{dx'}{dt}\vec{i'} + x'\frac{d\vec{i'}}{dt} + \frac{dy'}{dt}\vec{j'} + y'\frac{d\vec{j'}}{dt} + \frac{dz'}{dt}\vec{k'} + z'\frac{d\vec{k'}}{dt}$$

$$\Longrightarrow \underbrace{\frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k}} = \underbrace{\frac{dx'}{dt}\vec{i'} + \frac{dy'}{dt}\vec{j'} + \frac{dz'}{dt}\vec{k'}} + \underbrace{\frac{d\overrightarrow{OO'}}{dt} + x'\frac{d\vec{i'}}{dt} + y'\frac{d\vec{j'}}{dt} + z'\frac{d\vec{k'}}{dt}}$$

$\vec{\boldsymbol{V}}_{\boldsymbol{a}}(\boldsymbol{t})$: **Absolute Velocity** $\vec{\boldsymbol{V}}_{\boldsymbol{r}}(\boldsymbol{t})$ **: Relative Velocity** $\vec{\boldsymbol{V}}_{\boldsymbol{e}}(\boldsymbol{t})$ **: Training Velocity**

$$\boxed{\Rightarrow \vec{\boldsymbol{V}}_{\boldsymbol{a}}(\boldsymbol{t}) = \vec{\boldsymbol{V}}_{\boldsymbol{r}}(\boldsymbol{t}) + \vec{\boldsymbol{V}}_{\boldsymbol{e}}(\boldsymbol{t})}$$

Remark:

If the coordinate system R' is translational only with respect to R: $\vec{\boldsymbol{i'}}, \vec{\boldsymbol{j'}}, \vec{\boldsymbol{k'}} = \boldsymbol{Cst}$

$$\frac{d\vec{i'}}{dt} = \frac{d\vec{j'}}{dt} = \frac{d\vec{k'}}{dt} = 0 \Longrightarrow \vec{\boldsymbol{V}}_{\boldsymbol{e}}(\boldsymbol{t}) = \frac{\boldsymbol{d}\overrightarrow{\boldsymbol{OO'}}}{\boldsymbol{dt}}$$

Relationship between accelerations: $\vec{a}(t) = \frac{d\vec{V}}{dt}$

$$\frac{d}{dt}\left(\frac{dx}{dt}\vec{\imath} + \frac{dy}{dt}\vec{\jmath} + \frac{dz}{dt}\vec{k}\right) = \frac{d}{dt}\left(\frac{dx'}{dt}\vec{i'} + \frac{dy'}{dt}\vec{j'} + \frac{dz'}{dt}\vec{k'}\right) + \frac{d}{dt}\left(\frac{d\overrightarrow{OO'}}{dt} + x'\frac{d\vec{i'}}{dt} + y'\frac{d\vec{j'}}{dt} + z'\frac{d\vec{k'}}{dt}\right)$$

$$\frac{d^2x}{dt^2}\vec{\imath} + \frac{d^2y}{dt^2}\vec{\jmath} + \frac{d^2z}{dt^2}\vec{k} = \frac{d^2x'}{dt^2}\vec{i'} + \frac{dx'}{dt}\frac{d\vec{i'}}{dt} + \frac{d^2y'}{dt^2}\vec{j'} + \frac{dy'}{dt}\frac{d\vec{j'}}{dt} + \frac{d^2z'}{dt^2}\vec{k'} + \frac{dz'}{dt}\frac{d\vec{k'}}{dt}$$

$$+ \frac{d^2\overrightarrow{OO'}}{dt^2} + \frac{dx'}{dt}\frac{d\vec{i'}}{dt} + x'\frac{d^2\vec{i'}}{dt^2} + \frac{dy'}{dt}\frac{d\vec{j'}}{dt} + y'\frac{d^2\vec{j'}}{dt^2} + \frac{dz'}{dt}\frac{d\vec{k'}}{dt} + z'\frac{d^2\vec{k'}}{dt^2}$$

$$\vec{a}_a = \frac{d^2x}{dt^2}\vec{\imath} + \frac{d^2y}{dt^2}\vec{\jmath} + \frac{d^2z}{dt^2}\vec{k}$$ **: Absolute Acceleration**

$$\vec{a}_r = \frac{d^2x'}{dt^2}\vec{i'} + \frac{d^2y'}{dt^2}\vec{j'} + \frac{d^2z'}{dt^2}\vec{k'}$$ **: Relative Acceleration**

$$\boxed{\vec{a}_a = \vec{a}_r + \vec{a}_e + \vec{a}_C}$$

$$\vec{a}_e = \frac{d^2\overrightarrow{OO'}}{dt^2} + x'\frac{d^2\vec{i'}}{dt^2} + y'\frac{d^2\vec{j'}}{dt^2} + z'\frac{d^2\vec{k'}}{dt^2}$$ **: Training Acceleration**

$$\vec{a}_C = 2\left(\frac{dx'}{dt}\frac{d\vec{i'}}{dt} + \frac{dy'}{dt}\frac{d\vec{j'}}{dt} + \frac{dz'}{dt}\frac{d\vec{k'}}{dt}\right)$$ **: Coriolis acceleration**

Remarks: It is accepted that:

$$\frac{d\vec{i'}}{dt} = \vec{\omega} \wedge \vec{i'}\ , \frac{d\vec{j'}}{dt} = \vec{\omega} \wedge \vec{j'}\ , \frac{d\vec{k'}}{dt} = \vec{\omega} \wedge \vec{k'}$$

1- $$\vec{V}_e = \frac{d\overrightarrow{OO'}}{dt} + x'\frac{d\vec{i'}}{dt} + y'\frac{d\vec{j'}}{dt} + z'\frac{d\vec{k'}}{dt} = \frac{d\overrightarrow{OO'}}{dt} + x'\vec{\omega} \wedge \vec{i'} + y'\vec{\omega} \wedge \vec{j'} + z'\vec{\omega} \wedge \vec{k'}$$

$$= \frac{d\overrightarrow{OO'}}{dt} + \vec{\omega} \wedge \left(x'\vec{i'} + y'\vec{j'} + z'\vec{k'}\right) \quad \boxed{\Rightarrow \vec{V}_e = \frac{d\overrightarrow{OO'}}{dt} + \vec{\omega} \wedge \overrightarrow{O'M}}$$

2- $$\vec{a}_C = 2\left(\frac{dx'}{dt}\frac{d\vec{i'}}{dt} + \frac{dy'}{dt}\frac{d\vec{j'}}{dt} + \frac{dz'}{dt}\frac{d\vec{k'}}{dt}\right) = 2\left(\frac{dx'}{dt}\vec{\omega} \wedge \vec{i'} + \frac{dy'}{dt}\vec{\omega} \wedge \vec{j'} + \frac{dz'}{dt}\vec{\omega} \wedge \vec{k'}\right)$$

$$= 2\vec{\omega} \wedge \left(\frac{dx'}{dt}\vec{i'} + \frac{dy'}{dt}\vec{j'} + \frac{dz'}{dt}\vec{k'}\right) \quad \boxed{\Rightarrow \vec{a}_C = 2\vec{\omega} \wedge \vec{V}_r}$$

3- For $\vec{a}_e$: $$\frac{d^2\vec{i'}}{dt^2} = \frac{d}{dt}\left(\frac{d\vec{i'}}{dt}\right) = \frac{d}{dt}\left(\vec{\omega} \wedge \vec{i'}\right) = \frac{d\vec{\omega}}{dt} \wedge \vec{i'} + \vec{\omega} \wedge \frac{d\vec{i'}}{dt} = \frac{d\vec{\omega}}{dt} \wedge \vec{i'} + \vec{\omega} \wedge \left(\vec{\omega} \wedge \vec{i'}\right)$$

We replace in $\vec{a}_e$ and we find:

$$\boxed{\vec{a}_e = \frac{d^2\overrightarrow{OO'}}{dt^2} += \frac{d\vec{\omega}}{dt} \wedge \overrightarrow{O'M} + \vec{\omega} \wedge \left(\vec{\omega} \wedge \overrightarrow{O'M}\right)}$$